

Heave-pitch-roll Nonlinear Dynamics of a Spar Platform

Elvidio Gavassoni*, Paulo Batista Gonçalves**

* *Department of Civil Construction, Universidade Federal do Paraná, Curitiba, Brasil;* **
Department of Civil Engineering, Catholic University of Rio de Janeiro, Rio de Janeiro, Brasil.

Summary. Spars are stable floating platforms extensively used in the oil-gas industry. Such structures can experiment large amplitude motions under the actions of loads such as currents, winds, vortex-induced forces and waves. Such large-amplitude motion implies that a nonlinear dynamic analysis must be performed to capture complex phenomena. A feasible method to study the nonlinear dynamic phenomena is using the nonlinear normal modes [NNMs] theory. As an extension of the concept of linear normal mode, the NNMs could be used to reduce the order of the problem. Such reduced order models are used in this work to study the global coupling dynamic response to a heave-roll-pitch model of a spar platform under the action of wave and vortex-induced forces. First the nonlinear normal modes are analytical derived using the invariant manifold definition and the asymptotic method. The reduced order model is thus obtained, consisting of a single degree of freedom nonlinear oscillation for each NNM. These 1-DOF nonlinear oscillators are used to study the free and forced vibration of the spar and also to perform a parametric analysis of the problem. Important nonlinear phenomena are investigated such as multiple solutions, instabilities, multiplicity of modes, internal and parametric resonances. The results obtained using the reduced order models are compared to full integration of the coupled nonlinear equations of motion of the spar in order to verify the validity of the order reduction.

Introduction

Spars are suitable offshore platforms due to their adaptability to wide range of water depth and good motion performance [1][2]. Recently they have been used as floating offshore wind turbine[3] or wave energy converters [4]. A spar platform consists of a deep-draft and large-diameter cylinder that floats upright in the water and is characterized by a small water plane area and a large mass. As the use of these floating structures are pushed to deep water conditions they are becoming more susceptible to large amplitude motions due to offshore environmental loads. In this context a nonlinear analysis must be performed to anticipate some adverse conditions that could arise from complex nonlinear dynamic phenomena such internal and parametric resonances and instabilities that could threat the safe operational conditions of such structures. A full numerical nonlinear analysis of a spar vibration problem in offshore conditions requires very powerful computer resources and turn the design process a very cumbersome task [5]. A reduced order model could help offshore engineers in pre-design phases by offering insight on what nonlinear phenomena could appear in the vibration of spar floating platforms and for which parameter sets these phenomena are potentially dangerous to their structural integrity. The nonlinear normal modes could be used to derive such reduced order models. The nonlinear normal modes theory have been successfully used to study a large set of different practical problems including spar platforms[6]. The problem of heave-pitch coupling response of a spar under wave loads was studied by the authors using the nonlinear normal modes theory. However the inclusion of roll in the analysis is important to capture complex nonlinear phenomena such as roll-pitch parametric resonance. In this work the nonlinear coupled response of heave-roll-pitch three-degrees-of-freedom spar model under the action of wave excitation and vortex-induced force is investigated using a reduced order model based on the nonlinear normal modes theories. Nonlinear phenomena such as multiplicity of modes, unstable modes and solutions, multimodal motions are investigated through a parametric analysis. The stability of the solution is investigated by the application of the Floquet theory and Mathieu stability charts. Internal resonances are detected by singularities on the invariant manifold polynomial expansions. The results of the reduced order model obtained by the NNM approach are compared to the numerical solution of the original equations of motion. The good agreement between both solutions confirms the usefulness of the NNM in the nonlinear analysis of the spar platform and similar floating or compliant structures.

Structural model

The global dynamics of floating structures is generally modelled by rigid-body equations of motion [5]. This problem is governed by a six degrees of freedom system: three translational motions (x_1 – surge, x_2 – sway, and x_3 – heave); and three rotational motions (x_4 – roll, x_5 – pitch, and x_6 – yaw). The six degrees-of-freedom for the spar platform are shown in Figure 1. The system analysis can be reduced solely to the analysis of the heave (x_3), roll (x_4), and pitch (x_5) motions. This fact is due to the model symmetry, the relevance of the heave and pitch motions for the safe operation of the riser and mooring systems, and possible internal resonance associated to these types of motion. The first-order wave force is included in roll and pitch directions, while the vortex-induced force is considered in roll motion. The coupled nonlinear equations of motion can be written as:

$$\ddot{x}_3 + 2\mu_3\dot{x}_3 + \omega_3^2 x_3 - a_3 x_4^2 - a_3 x_5^2 = \bar{F}_3 \cos(\omega_w t); \quad (\text{eq.1-a})$$

$$\ddot{x}_4 + 2\mu_4\dot{x}_4 + \omega_4^2 x_4 - a_4 x_3 x_4 = \bar{F}_4 \cos(\omega_v t); \quad (\text{eq.1-b})$$

$$\ddot{x}_5 + 2\mu_5\dot{x}_5 + \omega_5^2 x_5 - a_5 x_3 x_5 = \bar{F}_5 \cos(\omega_w t); \quad (\text{eq.1-c})$$

where the dot above the variables represents its time derivatives, and:

$$\begin{aligned} \mu_3 &= \frac{B_3}{2(m+m_{33})}, \mu_i = \frac{B_i}{2(I+I_{ii})} \forall i=4,5, \omega_3^2 = \frac{\rho g A_w}{(m+m_{33})}; \omega_i^2 = \frac{\nabla \overline{GM}}{(I+I_{ii})} \forall i=4,5, \\ a_3 &= \frac{\rho g A_w H_g}{2(m+m_{33})}, a_i = \frac{(\nabla + 2\rho g A_w \overline{GM})}{2(I+I_{ii})} \forall i=4,5, \overline{F}_3 = \frac{F_3}{(m+m_3)}, \\ \overline{F}_i &= \frac{F_i}{(I+I_{ii})} \forall i=4,5 \end{aligned} \quad (\text{eq.2})$$

The parameters and variables definitions are showed in Table 1. The spar parameters used in this study are the same used in [7].

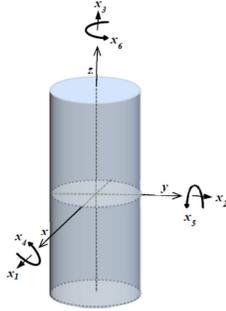


Figure 1 – Six degrees-of-freedom of a spar platform.

ρ	Sea water mass density	m_{33}	Spar added mass
A_w	Water plane area	I	Spar moment of inertia
g	Gravitational acceleration	I_{ii}	Spar added moment of inertia for roll (i=4) and pitch (i=5)
H_g	Distance from the centre of gravity to the undisturbed water surface	B_i	Radiation damping
\overline{GM}	Longitudinal metacentric height	$F_{3,5}$	First order wave exciting force for heave and pitch
∇	Displaced water volume	F_4	Vortex-induced force
m	Spar mass	ω_w, ω_v	Wave and vortex-induced force frequencies respectively

Table 1 – Spar parameters.

Due to the geometrical symmetry of the spar platform the roll and pitch coefficients a_4 and a_5 are approximated equal and thus the respective natural frequencies ($\omega_4=\omega_5$). Using this symmetries and by including the new dependent variables:

$$x_3 = \frac{\xi_3}{a_4} = \frac{\xi_3}{a_5}; x_4 = \frac{\xi_4}{\sqrt{a_3 a_4}}; x_5 = \frac{\xi_5}{\sqrt{a_3 a_5}}; \quad (\text{eq.3})$$

and the following dimensionless frequency parameter:

$$\varepsilon = \frac{\omega_3}{\omega_5} = \frac{\omega_3}{\omega_4}; \quad (\text{eq.4})$$

one obtains the set of equations of motion in the new variables spaces:

$$\ddot{\xi}_3 + 2\mu_3 \dot{\xi}_3 + \varepsilon^2 \omega_5^2 \xi_3 - \xi_4^2 - \xi_5^2 = a_4 \overline{F}_3 \cos(\omega_w t); \quad (\text{eq.5-a})$$

$$\ddot{\xi}_4 + 2\mu_4 \dot{\xi}_4 + \omega_5^2 \xi_4 - \xi_3 \xi_4 = \sqrt{a_3 a_5} \overline{F}_4 \cos(\omega_v t); \quad (\text{eq.5-b})$$

$$\ddot{\xi}_5 + 2\mu_5 \dot{\xi}_5 + \omega_5^2 \xi_5 - \xi_3 \xi_5 = \sqrt{a_3 a_5} \overline{F}_5 \cos(\omega_w t). \quad (\text{eq.5-c})$$

Nonlinear normal modes

The nonlinear normal mode (NNM) invariant manifold based definition [8] is used to derive reduced order models of the spar vibration problem. The NNMs are, using this definition, an invariant manifold surfaces that pass through the stable equilibrium configuration of the problem and are tangent to the underlining linear modes at this equilibrium point. The system of equations of motion (eq. 5) is rewritten in the Hamiltonian form:

$$\{\dot{\xi}\} = \{y\}, \{\dot{y}\} = \{f\}; \quad (\text{eq.6})$$

where ξ_i are the generalized displacements and y_i are the corresponding velocities. The generalized force vector $\{f\}$ consists of nonlinear moments that in general depend upon $\{\xi\}$ and $\{y\}$.

Each nonlinear mode motion can be entirely parameterized in terms of a single pair of generalized coordinates of the system, the displacement and velocity of a given degree-of-freedom, which is called master pair. Since the equations of motion (5) are uncoupled in the linear terms, each degree-of-freedom is chosen to derive its correspondent nonlinear mode. As an example the heave degree-of-freedom displacement-velocity pair, ξ_3 e y_3 , is chosen as the master pair:

$$u = \xi_3, v = y_3; \quad (\text{eq.7})$$

The other slave pairs (pitch and roll in the present example) are represented in terms of u and v by the constraint functions (P_i, Q_i):

$$\xi_i = P_i(u, v), y_i = Q_i(u, v) \rightarrow i = 3, 4, 5; \quad (\text{eq.8})$$

where, particularly, $P_3(u, v) = u$, $Q_3(u, v) = v$.

The next step is to eliminate the explicit time dependence of the equations. This is done by performing the time derivatives of the constraint equations. By replacing the time derivative resulting from the chain rule into the equations of motion (5) and by the use of the master pair definitions (7) and the slave relations (8), the following system of four second order partial differential equations is obtained:

$$Q_i(u, v) = \frac{\partial P_i(u, v)}{\partial u} v + \frac{\partial P_i(u, v)}{\partial v} f_1(u, P_2(u, v), \dots, P_n(u, v); v, Q_2(u, v), \dots, Q_n(u, v)) \rightarrow i = 4, 5 \quad (\text{eq.9})$$

$$f_i(u, P_2(u, v), \dots, P_n(u, v); v, Q_2(u, v), \dots, Q_n(u, v)) = \frac{\partial Q_i(u, v)}{\partial u} v + \quad (\text{eq.10})$$

$$\frac{\partial Q_i(u, v)}{\partial v} f_1(u, P_2(u, v), \dots, P_n(u, v); v, Q_2(u, v), \dots, Q_n(u, v)) \rightarrow i = 4, 5$$

The determination of the restraint equations leads to an order reduction of the problem, since their substitution in the original equations of motion results in a single degree-of-freedom nonlinear modal oscillator. Except for a few special cases, there is no closed exact solution for the differential partial equations that govern the invariant manifold expressed by (9) and (10). The solution can be derived in an analytical form using a Taylor series around the equilibrium configuration, taken here for simplicity as $\{\xi_0\} = 0$. Accordingly, the constraint functions can be written up to third order terms as:

$$P_i(u, v) = a_{1i}u + a_{2i}v + a_{3i}u^2 + a_{4i}uv + a_{5i}v^2 + a_{6i}u^3 + a_{7i}u^2v + a_{8i}uv^2 + a_{9i}v^3 \rightarrow i = 3, 4, 5; \quad (\text{eq. 11})$$

$$Q_i(u, v) = b_{1i}u + b_{2i}v + b_{3i}u^2 + b_{4i}uv + b_{5i}v^2 + b_{6i}u^3 + b_{7i}u^2v + b_{8i}uv^2 + b_{9i}v^3 \rightarrow i = 3, 4, 5 \quad (\text{eq. 12})$$

The substitution of equations (11) and (12) into the equation of motion results in an algebraic system of equations in terms of the constraint equations coefficients a_{ij} and b_{ij} and they can be sequentially solved. The so-obtained solution is valid only locally [9], and the validity domain is not known a priori, being determined only by comparison with numerical solutions of the original problem.

Nonlinear modal analysis

The substitution of the constraint equations into the original free undamped equations of motion results in a one-degree-of-freedom nonlinear oscillator, called modal oscillator and the standard well-known techniques developed to analyse 1-DOF oscillators can be used to study the dynamics of the problem and the slave-pair motions can be recovered using the restraint equations.

The modal oscillator for the heave mode is equal to:

$$\ddot{u} + \omega_3^2 u = 0 \quad (\text{eq. 13})$$

The resulting modal oscillator of the similar mode is linear. This result indicates that this is a fundamentally linear similar mode. No consistent additional mode is obtained using the master pair given by equation (7). Physically this means that a purely vertical spar motion can occur, without any rotation. This essentially translational motion is described, according to the hypothesis here adopted, by the linear equation of motion (13).

Next the generalized roll motion coordinates are used as the master pair and those related to heave and pitch as the slave pair. No similar mode is found using this master pair. The following nonsimilar mode is identified:

$$\ddot{u} + \omega_4^2 u - \frac{(2 - \varepsilon^2)u^3}{\omega_4^2 \varepsilon^2 (\varepsilon^2 - 4)} + \frac{2ui^2}{\omega_4^4 \varepsilon^2 (\varepsilon^2 - 4)} = 0 \quad (\text{eq. 14})$$

For the pitch motion coordinates one obtains:

$$\ddot{u} + \omega_5^2 u - \frac{(2 - \varepsilon^2)u^3}{\omega_5^2 \varepsilon^2 (\varepsilon^2 - 4)} + \frac{2ui^2}{\omega_5^4 \varepsilon^2 (\varepsilon^2 - 4)} = 0 \quad (\text{eq. 15})$$

Since the roll and pitch natural frequencies are identical due to physical symmetrical features of the spar, the roll (14) and pitch (15) equations are identical and it is sufficient to analyse the motion in one of these nonlinear normal modes to describe their nonlinear behaviour. In the case of 1:2 internal resonance ($\varepsilon=2$) the coefficients of the nonlinear terms of both 1-DOF oscillators will result in singularities indicating that the invariant approach cannot be used and a multimode analysis (using both heave and roll (or pitch) generalized coordinates as master pairs) must be performed, which, in the present case, result in 2-DOF order reduction of the problem instead of the 1-DOF obtained here. This results from the fact that the internal resonance is a violation of the invariance characteristic of the manifold of the

modelled (master) model [10]. Such feature of the NNM analysis permits to detect internal resonances. This is important in systems with a high number of degrees-of-freedom, where possible internal resonances are not known a priori. Both nonlinear single degree-of-freedom models have odd nonlinearities. In addition to the internal resonance, equations (14) and (15) show that the cubic geometric nonlinearity disappears when $\varepsilon = \sqrt{2}$. The sign of the cubic nonlinear terms are opposite which means that the system can exhibits a softening or a hardening behaviour according to the value of ε while oscillating in the roll and pitch mode.

The spar nonlinear oscillators (13-15) corresponds to an order reduction of the problem and they are used to derive important vibration features such as time response, frequency-amplitude in an analytical form, which are helpful information in the design phases of such offshore structures.

The modal lines derived from the time results for each modal oscillator are shown in Figure 3, the ξ_i coordinates are divided to ω_s^2 in order to facilitate the parametrical analysis. Figure 3(a) shows the heave similar mode and 3(b) the pitch and roll nonsimilar modes. One can note from Fig. 2 (b) that the modal curves are parabolic and they are influenced by the relation between the heave and roll (pitch) natural frequencies given by the ε parameter. Those curves have an horizontal tangent at point (0,0) in the configuration space due to the fact that the original equations of motion (5) are uncoupled in the linear terms.

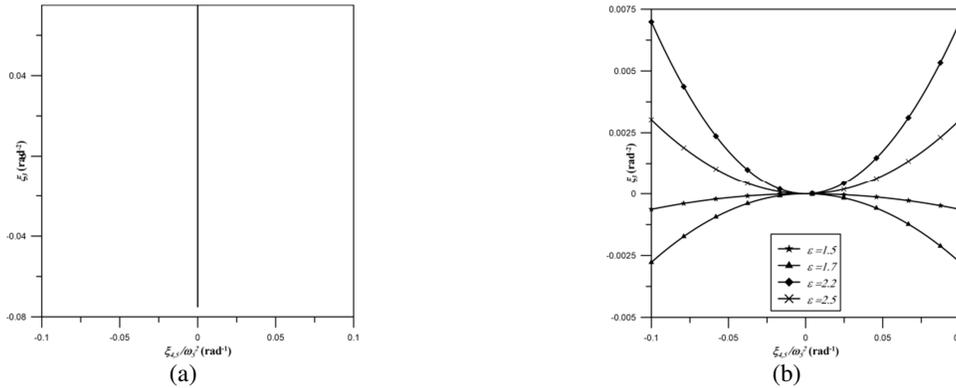


Figure 2 – Modal Curves: a) Heave mode; b) Roll and Pitch modes.

The frequency-amplitude curves can be analytically obtained for the three modes using the harmonic balance method applied to their respective modal oscillators, assuming the following approximated solution, in face of the observed nonlinearities in the modal oscillators:

$$u(t) = X \sin(\omega t) \quad (\text{eq. 16})$$

For the pitch (and roll mode) one obtain the following relation between the amplitude and frequency:

$$X^* = \frac{X}{\omega_s^2} = \pm \frac{2\varepsilon \sqrt{(2\Omega^2 - 3\varepsilon^2 + 6)(-4\Omega^2 + \Omega^2\varepsilon^2 - \varepsilon^2 + 4)}}{2\Omega^2 - 3\varepsilon^2 + 6}; \quad (\text{eq. 17})$$

where Ω is frequency dimensionless parameter defined as:

$$\Omega = \frac{\omega}{\omega_s} \quad (\text{eq. 18})$$

By studying the signal of the radicand of expression (17) one can determines where the spar has a softening or hardening behaviour according to the values of ε , this analysis is done using the chart presented in Figure 3. The curve $\varepsilon = 1/3(18 + 6\Omega^2)^{1/2}$ marks the sign alternation of the resultant of the cubic terms of (15), for ε values between zero and $(8/3)^{1/2}$, as the showed example where $\varepsilon = 0.5$, the spar motion on the heave mode has a softening behaviour. On the other side for ε values between $(8/3)^{1/2}$ and 2 (internal resonance - exclusively) the only possible solution to the spar while vibrating in the pitch mode presents a hardening behaviour (the curve for $\varepsilon = 1.75$ in Fig.3), above $\varepsilon = 2$ the solution presents again a softening behaviour (the showed curve for $\varepsilon = 2.5$ in Fig. 3).

The degree of agreement between the solution using the modal oscillators with the numerical solution up to some values of amplitudes can be used to reveal the validity region of the asymptotic method used to construct the reduced order model as showed in Figure 4 for $\varepsilon = 3.0$ and $\varepsilon = 1.75$ respectively. Fig. 4.a) shows a better agreement for larger amplitude values since case showed in Fig. 4.b) is very near to the internal resonance. The fidelity is small because it depends on the amplitude of motion as well as the proximity to internal resonances [11]. Far from the internal resonance region the asymptotical expansion has a better agreement as one can observe by comparing the phase portrait obtained using the modal oscillators and the numerical integration of the system of equations of motion. The

phase portraits are shown in Figure 5 for $\varepsilon=0.5$. The good agreement between both solutions, observed in Fig. 5, confirms the effectiveness of the reduced order model derived from the nonlinear modal analysis over the domain of validity of the series used in the approximated solution of the invariant manifold equations.

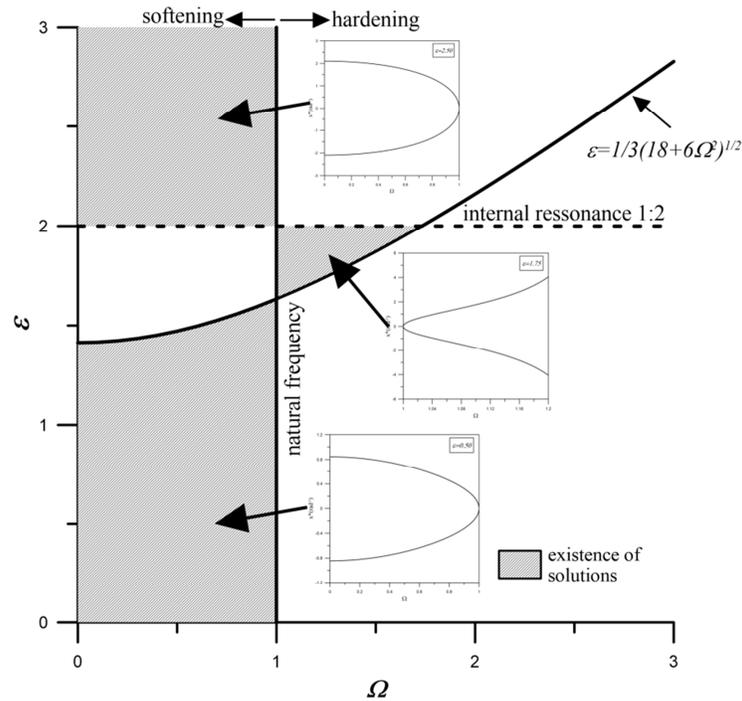


Figure 3 – Chart for the influence of ε on the frequency-amplitude curves for the pitch (roll) mode.

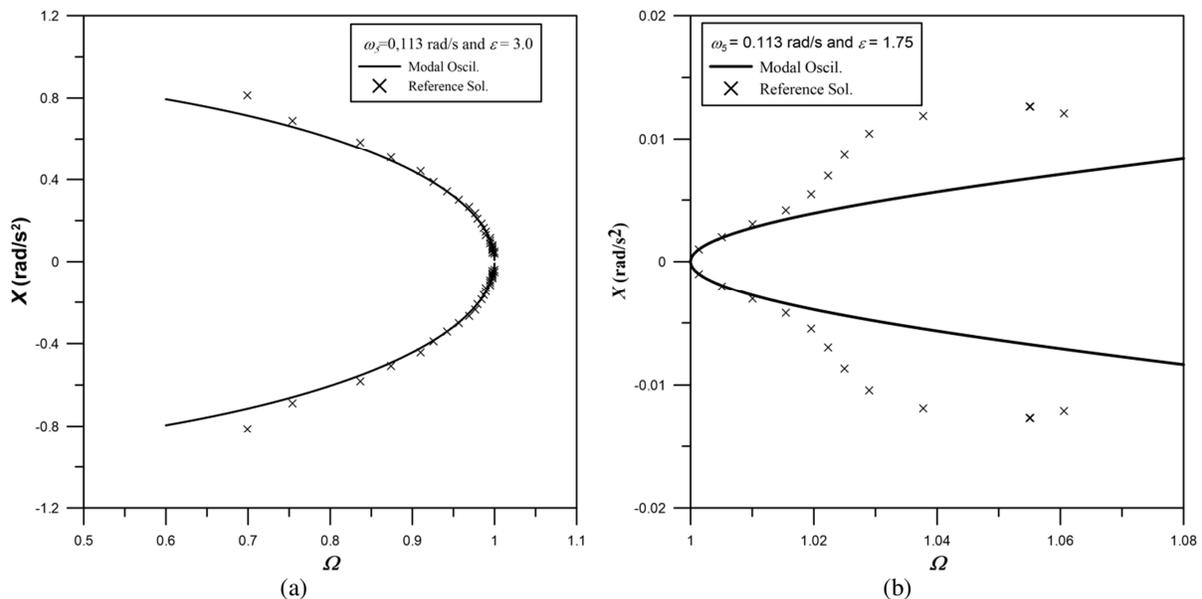


Figure 4 – Frequency-amplitude relations for pitch (roll) mode: a) softening behaviour and b) hardening behaviour.

Nonlinear multimodes

The internal resonance leads to a violation of the invariance of the one mode invariant manifold [12]. This fact demands a multimodal analysis. In order to generalize the invariant manifold based definition of nonlinear normal modes Boivin and co-workers introduced the concept of multi-mode invariant manifold, see [13] and [14]. The multi-mode manifold is constructed via the choice of M_m modes of interest, called modeled modes or master modes gathered in a subset S_m . The generalized coordinates of the master pairs are expressed by vectors $\{u_m\}$ and $\{v_m\}$

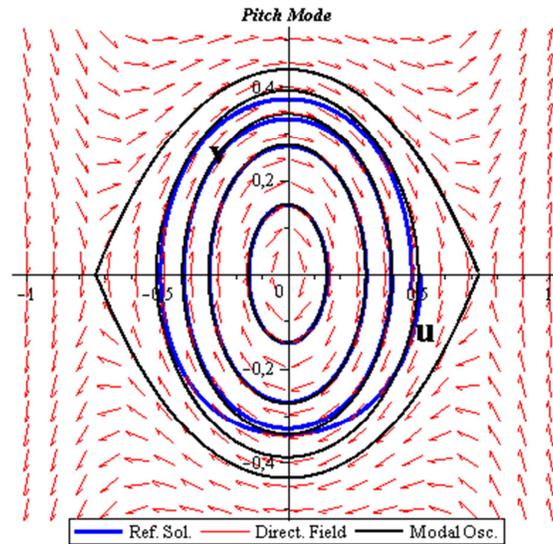


Figure 5 – Phase portrait for pitch (roll) mode.

respectively. The invariant manifold associated to the modeled modes has a dimension equal to $2M_m$ within the $2n$ space of the system with n degrees of freedom. The parameterization of the invariant manifold is done by using the M_m pairs of displacements and velocities:

$$\xi_i = u_i, y_i = v_i \quad \forall i \in S_M \quad (\text{eq. 19})$$

The slave coordinates are written in terms of the following constraint equations:

$$\xi_i = P_i(\{u_M\}, \{v_M\}), y_i = Q_i(\{u_M\}, \{v_M\}) \quad \forall i \notin S_M \quad (\text{eq. 20})$$

By the substitution of (19) and (20) in the equations of motion (6), the following partial differential equations governing the multi-mode invariant manifold are obtained:

$$Q_i = \sum_{k \in S_M} \frac{\partial P_i}{\partial u_k} v_k + \frac{\partial P_i}{\partial v_k} f_k \quad (\text{eq. 21})$$

$$f_i = \sum_{k \in S_M} \frac{\partial Q_i}{\partial u_k} v_k + \frac{\partial Q_i}{\partial v_k} f_k \quad (\text{eq. 22})$$

The asymptotical method using Taylor series about the S_M master pair coordinates are used to obtain the approximated analytical solution of equations (21) and (22). The substitution of the constraint functions in the equations of motion reduces the problem in an M -DOF multi-modal oscillator.

Nonlinear multi-modal analysis

In order to study the case of internal resonance (1:2) between the heave and pitch mode given by $\varepsilon=2$ it is necessary to retain the heave and pitch (or roll) as modeled modes in the derivation of the multi-mode manifold, i.e.:

$$\xi_i = u_i, y_i = v_i \quad \forall i = 3, 5 \in S_M \quad (\text{eq. 23})$$

The reduced order model has 2-DOF (heave and pitch) and is given by the following system of equations of motion:

$$\ddot{u}_3 + \varepsilon^2 \omega_3^2 u_3 - u_5^2 = 0 \quad (\text{eq.24-a})$$

$$\ddot{u}_5 + \omega_5^2 u_5 - u_3 u_5 = 0 \quad (\text{eq.24-b})$$

Both equations (24) have just quadratic nonlinearities and the system is uncoupled in the linear terms. If the roll is considered a modelled mode instead of the pitch mode, the equations of motion of the 2-DOF reduced order model are identical, due to the physical symmetries of the spar platform.

The numerical integration of (24) can be used to construct the phase portrait for heave and pitch sections in the multi-mode manifold vibration of the spar when the internal resonance occurs ($\varepsilon=2$) as showed in Figure 6. Both the heave (a) and pitch (b) sections show a center around the origin. The modal line for pitch and heave coordinates are showed

in the configuration subspace showed in Figure 7, since there is no linear correspondence to the multi-mode, the modal curve is not tangent at the origin as in the case of the individual modes showed in Fig.2.

The harmonic balance method can be used to obtain the frequency-amplitude relations to the 2-DOF (24) by using the following expressions

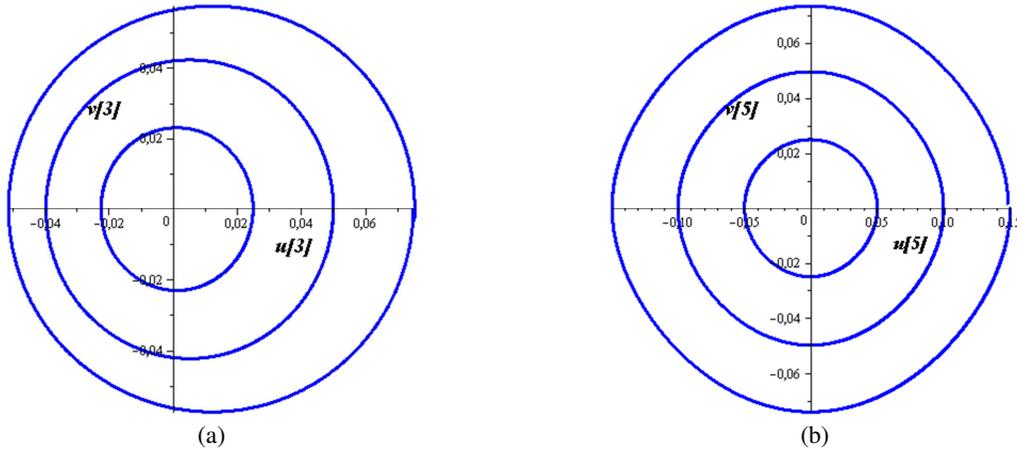


Figure 6 – Phase portrait for the multi-mode ($\epsilon=2$): a) heave and b) pitch.

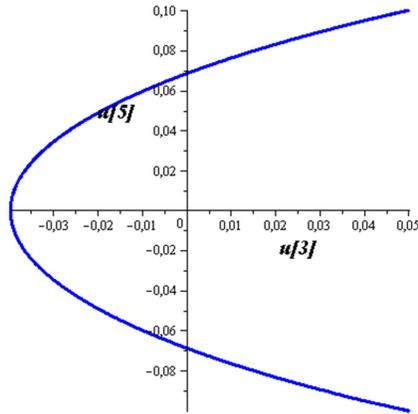


Figure 7 – Configuration space for the heave-pitch multi-mode ($\epsilon=2$).

$$u_3(t) = A + B \cos(\omega t); \tag{eq. 25-a}$$

$$u_5(t) = C + D \cos(\omega t); \tag{eq. 25-b}$$

and using the following expressions:

$$X_3^* = \frac{\sqrt{A^2 + B^2}}{\omega_5^2}; \quad X_5^* = \frac{\sqrt{C^2 + D^2}}{\omega_5^2}; \tag{eq. 26}$$

One can observe that the both heave and pitch motions present a softening behaviour in the case of the internal resonance 1:2 as showed in Figure 8, while the amplitude of pitch motions are higher than the heave amplitudes.

Conclusions

A reduced order model to study the nonlinear motion of a spar platform was derived. The model reduction was done using the nonlinear normal modes of the system. The spar has one similar mode and two nonsimilar modes. The motion in the pitch and roll nonlinear modes could present a hardening behaviour and a softening behaviour depending the relation between the natural frequencies of the heave and pitch (roll) motions. The system presents an internal resonance 1:2 between the heave and pitch motion. The multi-mode invariant manifold was used to study the motion in the internal resonance case. The results of the reduced model are compared to full numerical integration of the equations of motion which confirms the efficiency of the use of NNM-based reduced order methods in the studied examples. The reduced order models also allow fast parametric analysis which can be used in the pre-design phases.

This is a work in progress. Further work will include the forced vibration analysis include the wave-induced and vortex-induced motions and a more precise ocean loads consideration (buoyancy, added mass, and current action).

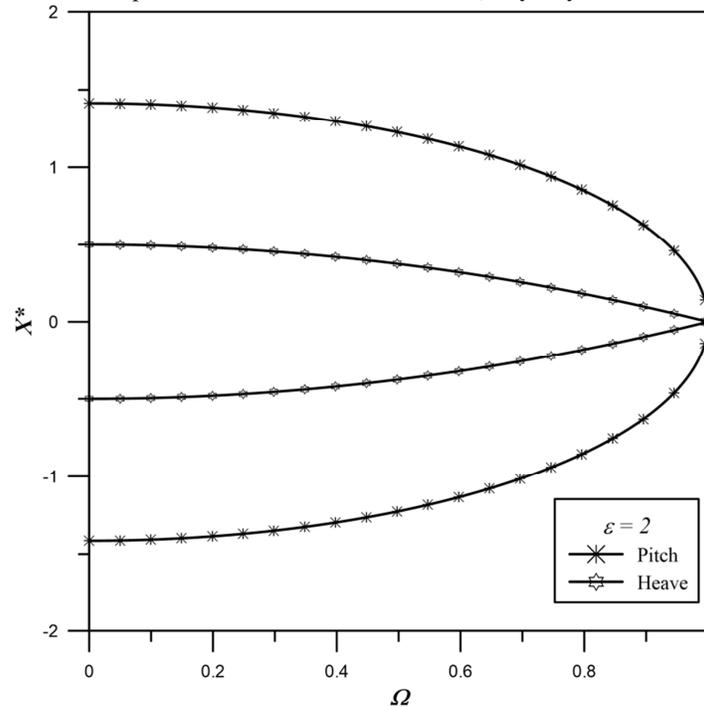


Figure 8 – Amplitude-Frequency relation at internal resonance ($\epsilon=2$).

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