Hopf bifurcation in a delayed nonlinear Mathieu equation

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Summary. We investigate the occurrence of Hopf bifurcations in the nonlinear Mathieu equation with delayed self-feedback. We include delay terms in the slow flow, extending previous work which replaced such terms by non-delayed terms. Our interest in this equation comes from its application to the dynamics of the synchrotron particle accelerator.

Introduction

In a paper by Sah and Rand [1], Hopf bifurcations in *autonomous* nonlinear oscillators with delayed self-feedback were studied using perturbation methods. It was shown that treating the slow flow as a DDE rather than an ODE, in general, gives a better approximation for the Hopf curves. The critical time delay T_{cr} for a Hopf bifurcation was expressed as a geometric series, giving a closed form expression for T_{cr}. In the present work, we apply the same procedure to a system studied by Morrison and Rand [2], who omitted delay terms in the slow flow. We ask, can the approximation be improved by including delay terms in the slow flow?

The delayed nonlinear Mathieu equation considered in this work has the following form:

$$\ddot{x} + (\delta + \epsilon\,\alpha\,\cos t)\,x + \epsilon\,\gamma\,x^3 = \epsilon\,\beta\,x(t-T) \tag{1}$$

Our interest in this equation comes from its application to the dynamics of the synchrotron particle accelerator. In this machine, magnetic fields are used to make a particle beam travel around a nearly circular path. The nonuniformity of the magnetic field leads to the appearance of the $\cos t$ term in the differential equation. As a particle in the beam traverses the path, it runs into its own wake which is modeled as delayed self-feedback.

Analysis

The method of two variable expansion posits that the solution depends on two time variables, $x(\xi,\eta)$, where $\xi = t$ and $\eta = \epsilon t$, such that at order ϵ^0 the solution takes the form:

$$x_0(\xi,\eta) = A(\eta)\cos\frac{\xi}{2} + B(\eta)\sin\frac{\xi}{2} \tag{2}$$

where we have set $\delta = \frac{1}{4} + \delta_1\epsilon + O(\epsilon^2)$. Carrying out the method, we end up with a delayed slow flow, which after linearization about the origin $(A,B) = (0,0)$ yields [2]:

$$\frac{dA}{d\eta} = \left[-\frac{\alpha}{2} + \delta_1\right]\,B - \beta\,A_d\sin\frac{T}{2} - \beta\,B_d\,\cos\frac{T}{2} \tag{3}$$

$$\frac{dB}{d\eta} = \left[-\frac{\alpha}{2} - \delta_1\right]\,A + \beta\,A_d\cos\frac{T}{2} - \beta\,B_d\,\sin\frac{T}{2} \tag{4}$$

We set

$$A = a\exp(\lambda\eta), \quad B = b\exp(\lambda\eta), \quad A_d = a\exp(\lambda\eta - \epsilon\lambda T), \quad B_d = b\exp(\lambda\eta - \epsilon\lambda T) \tag{5}$$

where a and b are constants. This gives

$$\begin{bmatrix} -\lambda - \beta\,\sin\frac{T}{2}\,\exp(-\epsilon\,\lambda\,T) & -\beta\,\cos\frac{T}{2}\,\exp(-\epsilon\,\lambda\,T) + \delta_1 - \frac{\alpha}{2} \\ \beta\,\cos\frac{T}{2}\,\exp(-\epsilon\,\lambda\,T) - \delta_1 - \frac{\alpha}{2} & -\lambda - \beta\,\sin\frac{T}{2}\,\exp(-\epsilon\,\lambda\,T) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{6}$$

For a nontrivial solution (a,b) we require the determinant to vanish:

$$\lambda^2 + 2\,\beta\,\sin\frac{T}{2}\,\lambda\,\exp(-\epsilon\,T\,\lambda) - 2\,\beta\,\delta_1\,\cos\frac{T}{2}\,\exp(-\epsilon\,T\,\lambda) + \beta^2\,\exp(-2\,\epsilon\,T\lambda) + \delta_1^2 - \frac{\alpha^2}{4} = 0 \tag{7}$$

For a Hopf bifurcation [3, 4], we set $\lambda = i\omega$ and use Euler's formula $\exp(-i\omega\epsilon T) = \cos\omega\epsilon T - i\sin\omega\epsilon T$. Separating real and imaginary parts we obtain

$$\beta^2\left(\sin\frac{T}{2}\right)^2\cos(2\epsilon T\omega)+\beta^2\left(\cos\frac{T}{2}\right)^2\cos 2\epsilon T\omega+2\beta\omega\sin\frac{T}{2}\sin\epsilon T\,\omega-2\delta_1\beta\cos\frac{T}{2}\cos\epsilon T\omega-\omega^2-\frac{\alpha^2}{4}+\delta_1^2=0 \quad (8)$$

$$-\beta^2\left(\sin\frac{T}{2}\right)^2\sin(2\,\epsilon\,T\,\omega)-\beta^2\left(\cos\frac{T}{2}\right)^2\sin 2\,\epsilon\,T\,\omega+2\,\delta_1\,\beta\,\cos\frac{T}{2}\,\sin\epsilon\,T\,\omega+2\,\beta\,\omega\,\sin\frac{T}{2}\,\cos\epsilon\,T\,\omega=0 \quad (9)$$

The next task is to analytically solve the two characteristic Eqs. (8)-(9) for the pair (ω,T). To this aim we use a perturbation schema by setting

$$\omega_{cr}=\sum_{n=0}^{N}\epsilon^n\,\omega_n=\omega_0+\epsilon\,\omega_1+\epsilon^2\,\omega_2+\ldots,\quad T_{cr}=\sum_{n=0}^{N}\epsilon^n\,T_n=T_0+\epsilon\,T_1+\epsilon^2\,T_2+\ldots \quad (10)$$

Inserting Eq. (10) in Eqs. (8)-(9), Taylor expanding the trig functions with respect to the small parameter $\epsilon << 1$, and equating terms of equal order of ϵ, we obtain the values of ω_n and T_n which are given as follows:

$$\omega_0=\frac{\sqrt{(2\,\beta-\alpha+2\,\delta_1)\,(2\,\beta+\alpha+2\,\delta_1)}}{2},\quad \omega_1=0,\quad \omega_2=-\frac{\delta_1\,\alpha^2\,\beta\,T_0^2}{8\,\omega_0},\quad \omega_3=\frac{\delta_1\,\alpha^2\,\beta\,(\beta+\delta_1)\,T_0^2}{2\,\omega_0},$$

$$T_0=2\,\pi,\quad T_1=-2\,(\beta+\delta_1)\,T_0,\quad T_2=4\,(\beta+\delta_1)^2\,T_0,\quad T_3=8\,(\beta+\delta_1)^3\,T_0-\frac{(\beta-2\,\delta_1)\,\left((\beta-\delta_1)^2-\omega_0^2\right)}{3}\,T_0^3$$

Note that the value of T_0 corresponds to the critical time delay obtained when the delayed variables A_d and B_d in Eqs. (3)-(4) are replaced by A and B resulting in an ODE slow flow, see [2].

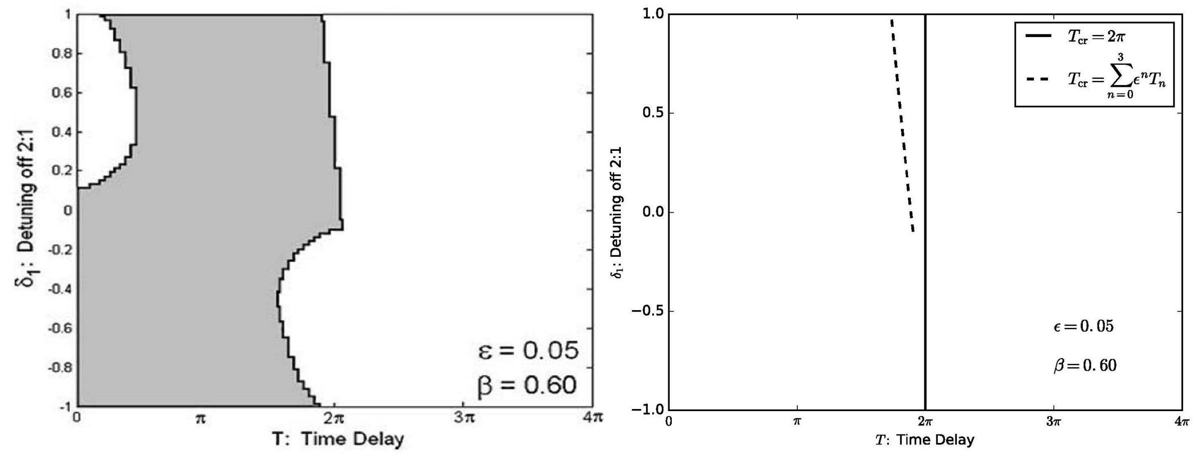

Figure 1: LEFT: Results of numerical integration of Eq.(1), from [2]. Shaded region is stable and unshaded region is unstable. The curved boundaries are saddle-node bifurcations, and are not dealt with in this paper. The straight portion of the boundary is a Hopf bifurcation and is to be compared to the perturbation results presented on the RIGHT side of this figure.
RIGHT: The solid line represents the Hopf curve obtained in [2] by replacing the delayed variables A_d and B_d in Eqs. (3)-(4) by A and B resulting in an ODE slow flow. The dashed line is the improved approximation obtained in this paper.

Conclusions

In this work we have analytically approximated Hopf bifurcation curves for the delayed nonlinear Mathieu equation (1). Fig. 1 shows a comparison between the results of (a) numerical integration, (b) the analytical critical time delay obtained in [2], $T_{\rm cr}=2\,\pi$, and (c) the one obtained using the procedure in this work, $T_{\rm cr}=\sum_{n=0}^{N}\epsilon^n\,T_n$. Comparison shows that keeping the delayed variables in the slow flow improves the analytical approximation.

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