Homoclinic orbits embedded in one-dimensional invariant manifolds of maps

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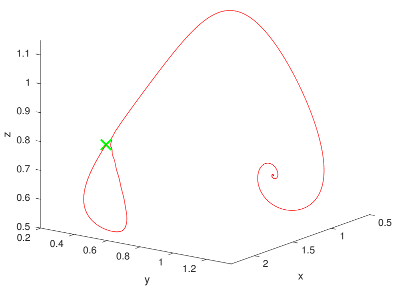
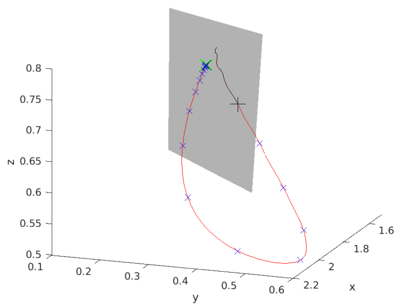
Summary. We describe new methods for initializing the computation of homoclinic orbits for maps in a state space with arbitrary dimension and for detecting their bifurcations. The initialization methods build on known and improved methods for computing one-dimensional stable and unstable manifolds. The methods are implemented in MatcontM, a freely available toolbox in Matlab for numerical analysis of bifurcations of fixed points, periodic orbits and connecting orbits of smooth nonlinear maps. The bifurcation analysis of homoclinic connections under variation of one parameter is based on continuation methods and allows to detect all known codimension 1 and 2 bifurcations in 3D maps, including tangencies and generalized tangencies. MatcontM provides a graphical user interface, enabling interactive control for all computations.
As the prime new feature we discuss an algorithm for initializing connecting orbits in the important special case where either the stable or unstable manifold is one-dimensional, allowing to compute all homoclinic orbits to saddle points in three-dimensional maps. We illustrate this algorithm in the study of the adaptive control map, a 3D map introduced in 1991 by Frouzakis, Adomaitis and Kevrekidis, to obtain a rather complete bifurcation diagram of the resonance horn in a 1:5 Neimark-Sacker bifurcation point, revealing new features.

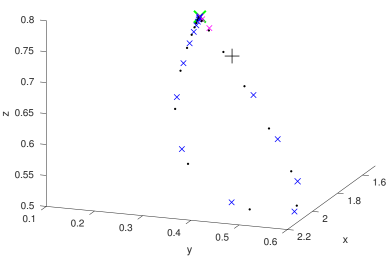
Computation of homoclinic orbits and tangencies in maps with dimension ≥ 3

To start a continuation of connecting orbits and tangencies we need a good initial orbit. We restrict to the case where one of the orbits (stable or unstable) is one-dimensional. We consider the *Adaptive Control Map* [1] a 3D map introduced in 1991 by Frouzakis, Adomaitis and Kevrekidis. We study a saddle fixed point of the fifth iteration with a one-dimensional unstable manifold.

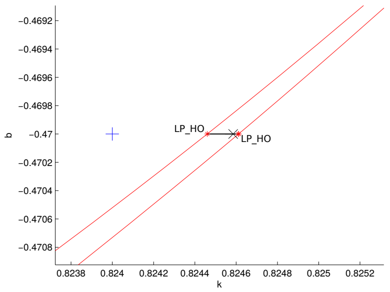
$$f : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} y \\ bx + k + zy \\ z - \frac{ky}{c+y^2}(bx + k + zy - 1) \end{pmatrix} \tag{1}$$

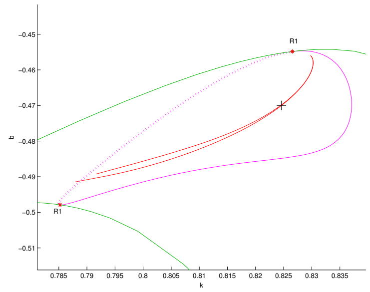


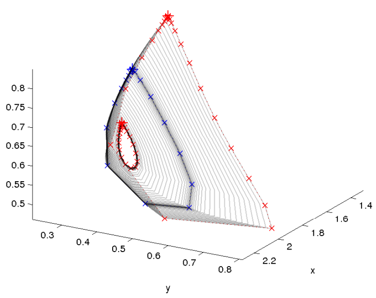




The left figure shows the unstable manifold emanating from the saddle, then advancing back towards it while showing slight oscillations before receding and moving on. This indicates that we are nearby a homoclinic connection. The middle figure shows the second step, we intersect the unstable manifold with the stable eigenspace (gray plane). The gray plane approximates the stable manifold nearby the saddle. We take the intersection closest to the saddle (black +). The black + is assumed to be part of the homoclinic orbit, we reconstruct the other points of the orbit (blue crosses). The right figure shows the constructed orbit (blue crosses) and the orbit after Newton corrections (black points).



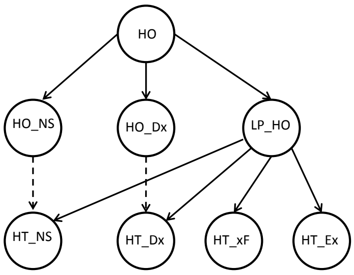




The left figure shows the continuation in the parameter space (black horizontal line) using the constructed orbit as an initial value. We vary the parameter k. The approximation is represented by the blue + (before correction) and the black cross (after correction). During continuation we detect homoclinic tangencies (LP_HO), we start homoclinic tangency continuations by varying parameters k and b (red lines). Robust homoclinic connections exist between those two red curves, which form a wedge. The left figure is a zoom of the middle figure which clearly shows a wedge. The green curves are limit point (fold) curves of the saddle and the magenta curve is a Neimark-Sacker (Hopf) curve of the saddle. The dashed magenta line is a Neutral Saddle curve. The wedge emanates from a resonance 1:1 bifurcation (R1). The right figure shows the homoclinic tangency continuation in the state space (blue orbit: starting tangency; smallest red orbit: homoclinic tangency near R1; largest red orbit: homoclinic tangency near the neutral saddles).

Bifurcations of homoclinic orbits and tangencies

We have designed test functions for detecting bifurcations during the continuation of homoclinic orbit and tangencies:



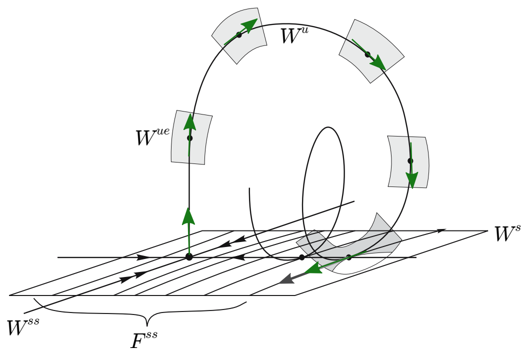
Homoclinic Orbit (HO)
Codimension 1 bifurcations:

- Neutral Saddle Homoclinic Orbit (HO_NS)
- Double-(Un)Stable Homoclinic Orbit (HO_Dx)
- Homoclinic Tangency LP_HO

Codimension 2 bifurcations:

- Neutral Saddle Homoclinic Tangency (HT_NS)
- Double-(Un)Stable Homoclinic Tangency (HT_Dx)
- (Un)Stable Foliation Homoclinic Tangency (HT_xF)
- Extended Stable Homoclinic Tangency (HT_ES)

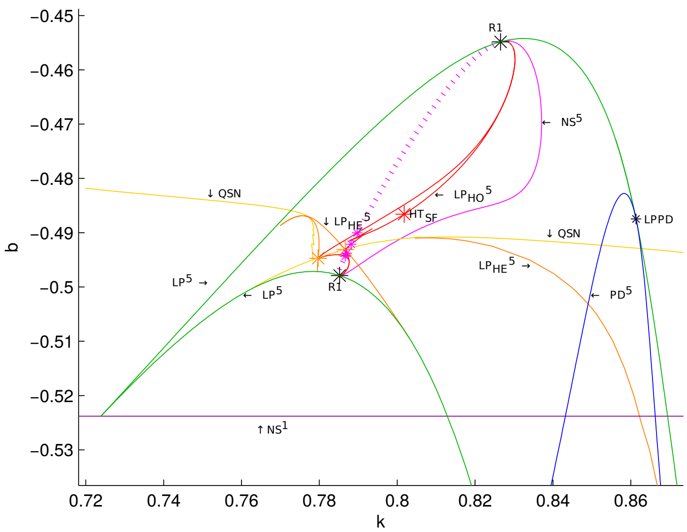
We highlight the *Stable Foliation Homoclinic Tangency bifurcation*: HT_SF. The tangent vector to the unstable manifold near the saddle is orthogonal to the leading stable left eigenvector which we assume to be real.



The green arrows are the tangent vectors of the unstable manifold. W^s, W^u, W^{ss} and W^{ue} denote the stable, unstable, strongly stable and unstable-extended manifolds, respectively. F^{ss} is the strongly stable foliation. Upon returning in the stable manifold the tangent vectors are tangential to the stable foliation. The Figure is adapted from [2].

Case study: 1:5 resonance horn

We apply our algorithms to generate a complete bifurcation diagram for (1) of a resonance horn in a 1:5 Neimark-Sacker bifurcation point. We detect and continue heteroclinic connections using similar strategies.



The horn (green) emanates from a Neimark-Sacker curve (dark magenta). All other curves are 5th iterate. The red curves are Homoclinic tangency curves (LP_HO), rooted in the R1 (one-to-one resonance) points. The orange curves are heteroclinic tangencies (LP_HE). Note the presence of a HT_SF bifurcation point (red star). We also detect four HT_NS bifurcations (magenta stars).

References

[1] C. E. Frouzakis, R. A. Adomaitis, and I. G. Kevrekidis. Resonance phenomena in an adaptively-controlled system. International Journal of Bifurcation and Chaos, 01:83-106, 1991.

[2] S. Gonchenko, V. Gonchenko, and J. Tatjer. Bifurcations of three-dimensional diffeomorphisms with non-simple quadratic homoclinic tangencies and generalized Hénon maps. Regular and Chaotic Dynamics, 12(3):233-266, 2007.