Model of Person Balancing on the Seesaw

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Summary. Relatively simple model of the process of maintaining balance by a person standing on a seesaw is considered. The model consists of a planar single-link inverted pendulum, which is connected by means of a cylindrical hinge, i.e., an “ankle joint”, to a support in the form of a segment of cylinder (seesaw), whose axis is perpendicular to the pendulum. The support can oscillate by rolling on a horizontal surface, and the pendulum can oscillate in the same plane as seesaw. The studied system has unstable equilibrium with vertical inverted pendulum and horizontal upper surface of the seesaw. The control is a torque applied on the axis of the hinge. This torque is assumed to have a restricted absolute value. A control law is constructed in the form of feedback along a single “unstable” coordinate of the open loop system in such a manner that the region of attraction of the unstable equilibrium would be the maximum possible. Several characteristic trajectories of the nonlinear system for the control constructed are considered.

Mathematical model of single-link pendulum on seesaw

We consider simple model of the maintenance of a human vertical posture on the movable platform (seesaw). The corresponding tests with persons standing on the seesaw have become popular in biomechanical investigations and sportive medicine [1-5]. It is natural to assume that under limited torque in the ankle the person standing on the seesaw strives to maximize the region of initial perturbations that can be overcome. Therefore, we study the problem of design control that *stabilizes* the unstable vertical posture and ensures “large” *basin* of attraction of this desired posture. The basin of attraction is understood to be the set of initial states from which the system asymptotically approaches the desired equilibrium. We study the mechanical system which consists of the seesaw of mass *m*, and inverted single-link pendulum of mass *M* fastened on its upper surface in point *S* by means cylindrical joint, *i.e.* “ankle joint”. The planar scheme of the considered mechanical model is shown in Fig. 1.

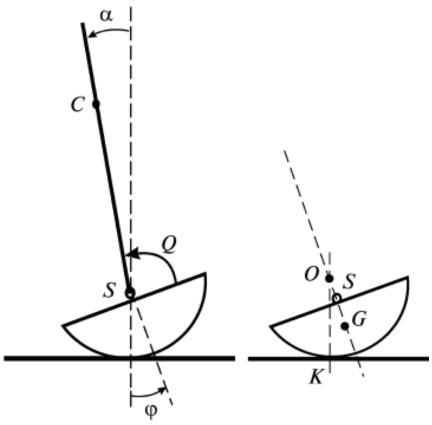

Figure 1.

Torque *Q* is applied in ankle joint *S*. Point *C* is the centre of mass of the pendulum, and ρ_p is its radius of inertia relative to point *S*. In fact, seesaw is the cylindrical segment. The cylinder radius is *R*, and ρ is the radius of inertia of the seesaw relative to its centre of mass *G* which is located in symmetry plane of the segment; point *O* is the axis of the cylinder. The seesaw is in contact with horizontal surface along the straight line *K*. It rolls over the surface without slipping and detaching. Straight lines *O*, *S*, and *K* are orthogonal to the plane of the drawing; *OK* is vertical line. Thus, the system has two degrees of freedom and two generalized coordinates – angles α and φ (see the left-hand part of Fig. 1). For the quantities shown in Fig. 1 we take the following notation and conditions: $OS = h > 0$, $OG = r > 0$, $SC = l$. In the first stage, it is expedient to consider the described above simplified model of the person oscillations. The results of the tests in [6] enable us to assume that the motion when the person is balancing on the seesaw, at least for some people, may be described approximately using a similar model.

The equations of motion in the form of Lagrange’s equations of the second kind are written in the following matrix form (*g* is the gravity acceleration):

$$A(q)\ddot{q}+F(q)\dot{q}^2+C\sin q=DQ\,,\quad q=\begin{Vmatrix}\alpha\\ \varphi\end{Vmatrix},\quad \dot{q}^2=\begin{Vmatrix}\dot{\alpha}^2\\ \dot{\varphi}^2\end{Vmatrix},\quad \sin q=\begin{Vmatrix}\sin\alpha\\ \sin\varphi\end{Vmatrix} \tag{1.1}$$

Here

$$A(q)=\begin{Vmatrix} M\rho_p^2 & Ml\left[R\cos\alpha-h\cos(\varphi-\alpha)\right]\\ Ml\left[R\cos\alpha-h\cos(\varphi-\alpha)\right] & M\left(R^2+h^2-2Rh\cos\varphi\right)+m\left(\rho^2+R^2+r^2-2Rr\cos\varphi\right)\end{Vmatrix},$$

$$F(q)=\begin{Vmatrix} 0 & Mlh\sin(\varphi-\alpha)\\ -Ml\left[R\sin\alpha+h\sin(\varphi-\alpha)\right] & (Mh+mr)R\sin\varphi\end{Vmatrix},\quad C=\begin{Vmatrix}-Mgl & 0\\ 0 & (Mh+mr)g\end{Vmatrix},\quad D=\begin{Vmatrix}1\\ -1\end{Vmatrix}$$

Motion equation (1.1) is written in matrix form in analogy to the equation describing in [7] the motion of an anthropomorphic mechanism.

It is obviously that if applied in the joint *S* torque $Q=0$, then system (1.1) has the equilibrium position

$$q=\dot{q}=0\ \left(\alpha=\varphi=\dot{\alpha}=\dot{\varphi}=0\right) \tag{1.2}$$

and this position is unstable.

Linearized dimensionless equations of motion, characteristic equation

Linearizing matrix equation (1.1) in the vicinity of the unstable equilibrium (1.2), we obtain

$$A_0\ddot{q}+Cq=DQ\,,\quad A_0=\begin{Vmatrix} M\rho_p^2 & Ml(R-h)\\ Ml(R-h) & m(R-r)^2+M(R-h)^2+m\rho^2\end{Vmatrix} \tag{2.1}$$

Introduce dimensionless time τ according to expression

$$\tau=t\sqrt{g/(R-h)} \tag{2.2}$$

Using prime $'$ to denote the differentiation of the variables with respect to dimensionless time (2.2), we write the linearized matrix equation (2.1) in the form

$$A_d q''+C_d q=Du\,,\quad A_d=\begin{Vmatrix}a & 1\\ 1 & b\end{Vmatrix},\quad C_d=\begin{Vmatrix}-1 & 0\\ 0 & c\end{Vmatrix} \tag{2.3}$$

Here

$$a=\frac{\rho_p^2}{(R-h)l}\,,\quad b=\frac{m(R-r)^2+M(R-h)^2+m\rho^2}{Ml(R-h)}\,,\quad c=\frac{Mh+mr}{Ml}\,,\quad u=\frac{Q}{Mgl} \tag{2.4}$$

Values *a*, *b*, and *c* are dimensionless parameters of the system, *u* is dimensionless control torque. It follows from expressions (2.4) that parameters *a* and *b* are positive, since $R>h$. Also parameter *c* is positive, since $h>0$ and $r>0$.

The characteristic equation of system (2.3) (with $u=0$) has the form

$$\sum_{n=0}^{4}e_n\lambda^{4-n}=0\,,\quad e_0=ab-1\,,\ e_1=0\,,\ e_2=ac-b\,,\ e_3=0\,,\ e_4=-c \tag{2.5}$$

Let us prove that coefficient for the highest power of characteristic Eq. (2.5) is positive, i.e.,

$$e_0=ab-1>0 \tag{2.6}$$

Inequality (2.6) is equivalent to the following inequality written in the original (dimensional) parameters:

$$\rho_p^2\left[m\left(R-r\right)^2+M\left(R-h\right)^2+m\rho^2\right]-Ml^2\left(R-h\right)^2>0 \qquad (2.7)$$

Recall that ρ_p is the radius of inertia of the pendulum relative to its fulcrum *S*, and *l* is the distance from fulcrum *S* to the centre of mass *C* of the pendulum; therefore,

$$\rho_p>l \qquad (2.8)$$

Under this condition (2.8) inequality (2.7) holds, and inequality (2.6) consequently also holds.

Characteristic equation (2.5) is biquadratic one:

$$e_0\lambda^4+e_2\lambda^2+e_4=0 \qquad (2.9)$$

Equation (2.9) has one positive root $\lambda_1>0$, one negative – $\lambda_2<0$, and two imaginary roots $\lambda_3,\lambda_4=\pm i\omega$, because $e_0>0$ and $e_4<0$. Besides, $\lambda_1=-\lambda_2$. The spectrum of the system is symmetrical relative to the both axes of the coordinates in the complex plane. And it is naturally because our system is conservative one.

Controllability of the system

To analyze the controllability of system (2.1) (or (2.3)) in Kalman sense we use Hautus criterion [8-10] for second-order systems. In accordance with this criterion, system (2.3) is fully controllable, if and only if the following equality holds for all the eigenvalues λ of this system:

$$\operatorname{rank}\left\|A_d\lambda^2+C_d,\quad D\right\|=2 \qquad (3.1)$$

For matrices A_d, C_d and *D* relation (3.1) takes the following form:

$$\operatorname{rank}\left\|\begin{matrix}a\lambda^2-1 & \lambda^2 & 1\\ \lambda^2 & b\lambda^2+c & -1\end{matrix}\right\|=2 \qquad (3.2)$$

Since the eigenvalues λ of the system satisfy characteristic equation (2.9), for controllability it is necessary and sufficient that for each eigenvalue of the system, at least one of the two matrices

$$\left\|\begin{matrix}a\lambda^2-1 & 1\\ \lambda^2 & -1\end{matrix}\right\| \quad \text{or} \quad \left\|\begin{matrix}\lambda^2 & 1\\ b\lambda^2+c & -1\end{matrix}\right\| \qquad (3.3)$$

appearing from relation (3.2) would have full rank. This requirement is not satisfied if the determinants of both matrices in (3.3) vanish simultaneously when at least one of the roots of Eq. (2.9) is substituted into them. In other words, this requirement is not satisfied if the system relative to λ considered, which is composed of Eq. (2.9), and equations

$$\left(a+1\right)\lambda^2=1,\quad \left(b+1\right)\lambda^2=-c \qquad (3.4)$$

has at least one solution. From equations (3.4), after λ^2 is eliminated from them, we obtain the equality $(a+1)c+b+1=0$, but it cannot be satisfied, because *a*, *b* and *c* are positive values. Therefore, the system of equations (2.9) and (3.4) does not have any solutions. Thus, system (2.3) is fully Kalman-controllable.

Torque of the viscous friction forces

Let us represent torque *Q* as a sum of control torque Q_u developed by the drive and the torque of the acting in the ankle *S* viscous friction forces with coefficient $k>0$:

$$Q=Q_u-k(\dot{\alpha}-\dot{\varphi}) \qquad (4.1)$$

Using this last expression, equation (1.1) can be rewritten as follows:

$$A(q)\ddot{q}+B\dot{q}+F(q)\dot{q}^2+C\sin q=DQ_u,\quad B=\|b_{ij}\|\ \ (i,j=1,2),\quad b_{11}=b_{22}=k,\ \ b_{12}=b_{21}=-k \tag{4.2}$$

Below we consider the motion of system (4.2) under admissible control torque $Q_u \in PC$ that is restricted in absolute value:

$$|Q_u(t)| \le Q_u^0 \quad (Q_u^0 = const) \tag{4.3}$$

It is obviously that if torque $Q_u = 0$, then system (4.2) (as system (1.1)) has unstable equilibrium position (1.2). Linearizing equation (4.2) in the vicinity of the unstable equilibrium position (1.2), we obtain

$$A_0\ddot{q}+B\dot{q}+Cq=DQ_u \tag{4.4}$$

System (4.4) is Kalman-controllable as system (2.3) because the torque of the viscous friction forces is applied in the same ankle-joint as the control torque developed by the drive (see expression (4.1)).

Introducing dimensionless time τ according to formula (2.2) we come to the equation

$$A_d q''+\chi B_d q'+C_d q=Du,\qquad B_d=\begin{Vmatrix}1 & -1\\ -1 & 1\end{Vmatrix} \tag{4.5}$$

with the same matrices A_d, C_d as in expressions (2.3) and the same dimensionless parameters *a*, *b*, and *c* as in expressions (2.4). But in equation (4.5) (see inequality (4.3))

$$\chi=\frac{k}{Mgl},\quad u=\frac{Q_u}{Mgl}\quad \left(|u(\tau)|\le u^0,\ u^0=\frac{Q_u^0}{Mgl}=const\right) \tag{4.6}$$

Here value $\chi>0$ is dimensionless coefficient of viscous friction, *u* is dimensionless control action developed by the drive.

The characteristic equation of system (4.5) (for $u=0$) is described by expression (2.5) with the same coefficients e_0, e_2, e_4, but with

$$e_1=\chi(a+b+2\chi),\quad e_3=\chi(c-1) \tag{4.7}$$

We will examine the issue of the number of roots of Eq. (2.5) (see also coefficients (4.7)) that have a positive real part when there is friction in hinge *S*. In accordance with the Routh–Hurwitz criterion [11] the number of roots of algebraic equation (2.5) with a positive real part is equal to the number of sign changes along the sequence

$$G_0,\ G_1,\ G_1G_2,\ G_2G_3,\ e_4 \tag{4.8}$$

where

$$G_0=e_0>0,\ \ G_1=e_1>0,\ \ G_2=\begin{vmatrix}e_1 & e_0\\ e_3 & e_2\end{vmatrix},\ \ G_3=\begin{vmatrix}e_1 & e_0 & 0\\ e_3 & e_2 & e_1\\ 0 & e_4 & e_3\end{vmatrix},\ \ e_4=-c<0$$

We evaluate determinant G_2:

$$G_2=e_1e_2-e_0e_3=\chi\left[(a+b+2\chi)(ac-b)+(ab-1)(1-c)\right] \tag{4.9}$$

If the distance $l=SC$ is sufficiently large, then inequalities $c<1$ and $ac-b>0$ hold. The latter inequality follows from expressions (2.4) and inequality (2.8). When these conditions and condition (2.6) are satisfied, then, as follows from expression (4.9), $G_2>0$. Then $G_1G_2>0$. Now in sequence (4.8) there is exactly one sign change, regardless of the sign of G_3. In this case, Eq. (2.5) has only one root λ_1 located in the right half-plane of the complex plane. This root λ_1 is real one of course.

Design of the stabilization algorithm with large basin of attraction

On the basis of the model adopted, we will consider the problem of stabilizing small oscillations of a single-link inverted pendulum mounted on the seesaw. Let for $u = 0$ and $\chi > 0$ system (4.5) has a single eigenvalue λ_1, which is located in the *right half-plane* of the complex plane, while all of its other eigenvalues lie in the *left half-plane*. For such a system we want to construct a control, which stabilizes the unstable vertical position of the inverted pendulum and the horizontal position of the seesaw platform with the maximum basin of attraction. In other words, we want to stabilize the unstable equilibrium (1.2) with maximal as possible basin of attraction. Here the basin of attraction is understood to be the set of initial states from which the system asymptotically approaches the origin of coordinates. One of the natural problems of a person standing on a seesaw is to return to a state of balance when there are “large” deviations from it. In other words, it is natural to assume that the person strives to maximize the region of initial perturbations that can be overcome. To solve the formulated above problem we will use the method previously described in [12].

We write system (4.5) in the Cauchy form, i.e., in the form of the system of first-order equations

$$\mathrm{x}' = \Phi\mathrm{x} + Hu\,;\quad \mathrm{x} = \begin{Vmatrix} \mathrm{q} \\ \mathrm{q}' \end{Vmatrix},\quad \Phi = \begin{Vmatrix} 0 & I \\ -A_u^{-1}C_u & -\chi A_u^{-1}B_u \end{Vmatrix},\quad H = \begin{Vmatrix} 0 \\ A_u^{-1}D \end{Vmatrix} \tag{5.1}$$

Here I is the unit matrix. Equilibrium state (1.2) is described in the new variables by the relation

$$\mathrm{x} \equiv 0 \tag{5.2}$$

We reduce system (5.1) to Jordan variables using the non-degenerate transformation

$$y = Tx \tag{5.3}$$

and isolate the equation which corresponds to the positive eigenvalue λ_1 from it. This equation describes the behaviour of an “unstable” (in the absence of the control, $u = 0$) Jordan variable, which we will denote by y_1:

$$y_1' = \lambda_1 y_1 + pu \tag{5.4}$$

In Eq. (5.4) $p \neq 0$, since the original system is completely controllable in the Kalman sense. By choosing the sign of the variable y_1 we can ensure satisfaction of the inequality p > 0.

We will assign the control *u* in the form of linear feedback along the unstable coordinate y_1:

$$u = -\gamma y_1 \tag{5.5}$$

When $\gamma > \lambda_1 / p$, control (5.5) ensures asymptotic stability of the trivial solution $y_1 = 0$ of Eq. (5.4) and consequently of solution (5.2) of the entire system (5.1), since it does not alter (“does not shift”) the remaining three eigenvalues of this system, which have negative real parts.

Under restriction $|u| \le u^0$ (see (4.6)) linear control (5.5) takes the form of linear feedback control with saturation:

$$u = \begin{cases} -u^0 & \text{when} \quad \gamma y_1 \le -u^0 \\ -\gamma y_1 & \text{when} \quad \gamma |y_1| \le u^0, \qquad \gamma > \lambda_1 / p \\ u^0 & \text{when} \quad \gamma y_1 \ge u^0 \end{cases} \tag{5.6}$$

Control (5.6) guarantees the maximum possible basin of attraction for the trivial solution $y_1 = 0$ of Eq. (5.4), as well as for solution (5.2) of the entire system (5.1). The prove of this assertion is described in [12]. This basin is described in Jordan variables by the inequality

$$|y_1| < \frac{u^0 p}{\lambda_1} \tag{5.7}$$

Jordan variable y_1 is linear combination of phase variables $\alpha,\ \varphi,\ \dot{\alpha},\ \dot{\varphi}$. Consequently control (5.6) depends on these four original variables.

In dimensional variables control (5.6) has the form

$$Q_u = \begin{cases} -Q_u^0 & \text{when} \quad Q_u^* \le -Q_u^0 \\ Q_u^* & \text{when} \quad \left|Q_u^*\right| \le Q_u^0 \\ Q_u^0 & \text{when} \quad Q_u^* \ge Q_u^0 \end{cases} \tag{5.8}$$

Here

$$Q_u^* = -\gamma Mgl\left[T_{11}\alpha + T_{12}\varphi + \sqrt{\frac{R-h}{g}}\left(T_{13}\dot{\alpha} + T_{14}\dot{\varphi}\right)\right], \qquad \gamma > \lambda_1/p \tag{5.9}$$

and T_{1j} $\left(j = 1,\ 2,\ 3,\ 4\right)$ are the components in the first row of transformation matrix T (see expression (5.3)); this row corresponds to the variable y_1. Basin of attraction (5.7) is described in the original dimensional variables of state by the inequality

$$\left|T_{11}\alpha + T_{12}\varphi + \sqrt{\frac{R-h}{g}}\left(T_{13}\dot{\alpha} + T_{14}\dot{\varphi}\right)\right| < \frac{Q_u^0 p}{Mgl\lambda_1} \tag{5.10}$$

and has the form of a hyperlayer in the four-dimensional space of the phase variables $\alpha,\ \varphi,\ \dot{\alpha}$ and $\dot{\varphi}$. When $T_{1j} \ne 0$ $\left(j = 1,\ 2,\ 3,\ 4\right)$, this hyperlayer intersects the coordinate axes at the values of the phase coordinates

$$\alpha_{\sup} = \pm\frac{Q_u^0 p}{Mgl\lambda_1 T_{11}}, \quad \varphi_{\sup} = \pm\frac{Q_u^0 p}{Mgl\lambda_1 T_{12}}, \quad \dot{\alpha}_{\sup} = \pm\sqrt{\frac{g}{R-h}}\frac{Q_u^0 p}{Mgl\lambda_1 T_{13}}, \quad \dot{\varphi}_{\sup} = \pm\sqrt{\frac{g}{R-h}}\frac{Q_u^0 p}{Mgl\lambda_1 T_{14}}$$

Trajectories of motion

We take the following values of the parameters of the system:

$$M = 106\ kg, \quad m = 2.2\ kg, \quad l = 0.9\ m, \quad R = 0.45\ m, \quad h = 0.38\ m, \quad r = 0.41\ m, \quad \rho = 0.12\ m, \quad Q_u^0 = 49\ N \cdot m$$

These parameters correspond approximately to a man of normal constitution above average age. The radius of inertia $\rho_p = 1.04\ m$ was calculated for a homogeneous thin rod of length $2l$. The roots of characteristic equation (2.5) are equal to ± 0.23 and $\pm 4.25i$, when $k = 0$ $\left(\chi = 0\right)$ and to $-1.5 \pm 3.9i$, to -0.24, and 0.23, when $k = 5\ N \cdot m \cdot s$ $\left(\chi = 0.06\right)$, i.e., $\lambda_1 = 0.23$.

In Fig. 2, we present for $k = 5\ N \cdot m \cdot s$ the characteristic trajectories of the motion of our systems.

The shown in Fig. 2 trajectories were obtained as a result of numerical integration of complete nonlinear equations (4.2) and of linear equations (4.4) under initial conditions that are close to the boundaries of basin of attraction (5.10). The control was chosen in form (5.8), (5.9) with $\gamma = 3\lambda_1/2p$ $\left(p = 0.21\right)$. The initial conditions were chosen so that the angle of deviation α from the equilibrium position $\alpha = 0$ would be equal to 98% of the value of $\left|\alpha_{\sup}\right|$ and the angle of inclination of the seesaw φ and both angular velocities $\dot{\alpha}$ and $\dot{\varphi}$ would be equal to zero.

The upper and middle parts of Fig. 2 show the time dependences of the angle of deviation α of the pendulum from the vertical and the angle of rotation φ of the platform. The lower part of Fig. 2 shows the time dependence of the control torque Q_u. The solid lines show the results of the numerical solution of linearized system (4.4), and the dashed lines show the results of the solution of nonlinear system (4.2). Under the selected initial conditions, the trajectories of the nonlinear system (4.2) are close to the trajectories of the linear system (4.4) and both tend asymptotically to the equilibrium posture. According to the graphs presented, in the case considered the motion breaks down into two stages: in the first stage, torque Q_u takes the minimum possible value $-Q_u^0$, and the system moves near the equilibrium q_e, which satisfies the relation

$$C \sin q_e = -DQ_u^0$$

In this stage of the motion, the pendulum deviates by an angle close to the value

$$\alpha_e = \arcsin \frac{Q_u^0}{Mgl} \approx 0.052$$

and the seesaw turns “rapidly” through an angle close to the value

$$\varphi_e = \arcsin \frac{Q_u^0}{(Mh + mr)\, g} \approx 0.12$$

After this, the pendulum and the seesaw perform oscillations in the vicinity of these “intermediate” equilibrium. When the coefficient of friction is large, the oscillations decay rapidly in this stage. In the second stage, the system tends to the assigned equilibrium $q = 0$. The torque Q_u, following the variable y_1, asymptotically tends to zero.

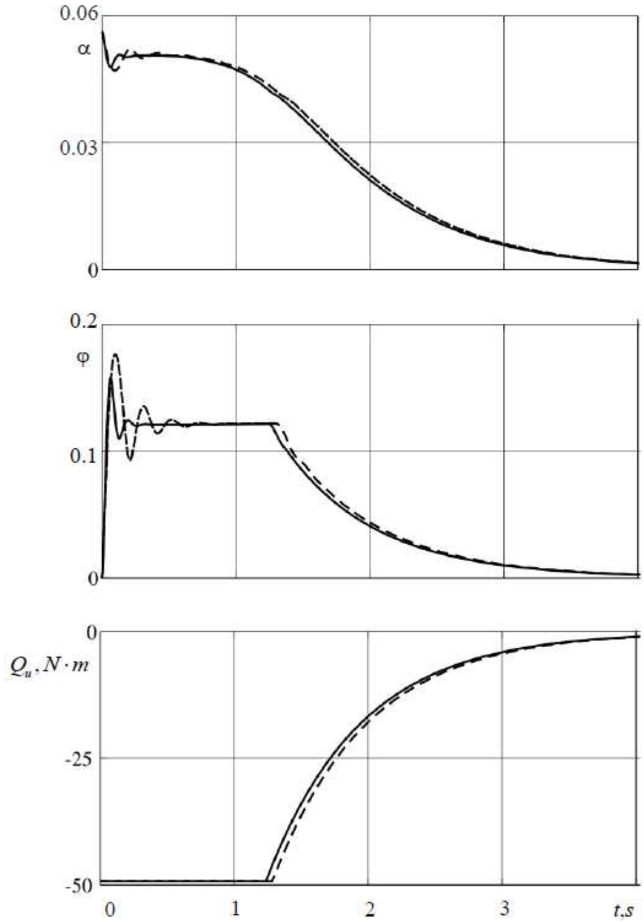

Figure 2.

At small values of the viscosity coefficient *k*, oscillations which “correspond” to complex eigenvalues decay slowly, but the character of the motion does not change.

The described above behavior of the system (4.2) under control (5.8), (5.9) corresponds to some extent to sensations of several test subjects when they tried hard to maintain balance on the seesaw.

Conclusion

Feedback control with saturation is designed to stabilize the inverted single-link pendulum on the seesaw. Designed control law ensures large basin of attraction of unstable equilibrium. Studied mechanical system can be considered as a model of a human maintaining balance on the seesaw.

Acknowledgements

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