

Cylindrical Cavity Evolution in a Plane Parallel Potential Flow of the Perfect Incompressible Fluid

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Summary. This manuscript proposes a numerical algorithm for the problem of cylindrical cavity evolution in a plane parallel potential flow of the perfect incompressible fluid. Based on the boundary elements method with no saturation, the algorithm has a high order of approximation and reasonable computation time. It is required that the distances between adjacent points on the boundary change in proportion to the length of the boundary. This requirement provides the numerical stability of the algorithm.

General discussion

Developing numerical algorithms for mechanical problems with a free boundary is one of the most common problems in numerical modelling. Cylindrical cavity evolution in a plane parallel potential flow of the perfect incompressible fluid is a problem of that kind. This manuscript proposes a numerical algorithm for the problem that is based on the boundary elements method.

Let us give basic mathematical notations and definitions required for the further discussion. Since the algorithm is based on the boundary elements method, key equations are defined on the boundary of a cylindrical cavity. Let us denote the boundary of a cavity by ∂S and the exterior domain by S . Let us also denote by Φ the potential function and by Ψ the stream function. Both of them depend on coordinates (x, y) and time t . Since a potential flow of the perfect incompressible fluid is considered, Φ and Ψ functions satisfy Laplace's equation in S . The complex potential W is defined by the following formula:

$$W = \Phi + i\Psi, \quad z = x + iy.$$

The simplest case of the problem is stated in paper [1]. It proposes a numerical algorithm for the problem of deformation of a circular cylindrical cavity with a unit radius in a plane parallel flow. The complex potential function in the initial time is defined by the following equation:

$$W = \frac{1}{z}$$

(the initial condition for the problem of flow past the circular cylindrical cavity moving with a unit velocity). Corresponding potential and stream functions are defined as follows:

$$\Phi = \frac{x}{x^2 + y^2}, \quad \Psi = -\frac{y}{x^2 + y^2}.$$

The paper provides some additional requirements. The cavity area is required to be constant. There is also the requirement imposed by the Bernoulli's equation.

The algorithm proposed in this manuscript satisfies the above requirements, it is more stable rather than the one proposed in paper [1] and allows us to improve the previous calculation results for the already stated and some other problems.

Basic equations

The main equation that lays a foundation for the algorithm is defined on the boundary ∂S of the cavity and expresses the relation between a potential function Φ and a stream function Ψ .

$$\begin{aligned}
 -2\pi\Psi &= A \frac{\partial\Phi}{\partial s} + B\Psi, \quad \text{where} & (1) \\
 \left[A \frac{\partial\Phi}{\partial s} \right] (M, t) &:= \oint_{\partial S} G(M, M') \frac{\partial\Phi}{\partial s'}(M', t) ds', \\
 [B\Psi] (M, t) &:= \oint_{\partial S} \frac{\partial G}{\partial n'}(M, M') (\Psi(M', t) - \Psi(M, t)) ds', \\
 G(M, M') &:= \ln(r(M, M')), \\
 r^2(M, M') &:= (x - x')^2 + (y - y')^2, \\
 M(x, y), M'(x', y') &\in \partial S.
 \end{aligned}$$

The algorithm proposed in [1] uses similar equation for the potential function Φ that has the following form:

$$-2\pi\Phi = -AV + B\Phi, \quad \text{where} \quad V := \frac{\partial\Phi}{\partial n}. \quad (2)$$

Plane-parallel flow requires only two dimensions x and y for computations, therefore, for any moment of time t , the cavity boundary has the dimension of one and can be parameterized with some ζ . Let us do it as follows:

$$ds = l(t)f(\zeta)d\zeta, \quad 0 \leq \zeta \leq 1, \quad \int_0^1 f(\zeta) d\zeta = 1, \quad (3)$$

where $l(t)$ is the overall length of the boundary and $f(\zeta)$ is a positive function that is used to control the distribution of points along the cavity boundary. Let us call it the density function.

The normal component of boundary velocity can be obtained from the following equation:

$$V(\zeta, t) = \frac{\partial \Psi}{\partial s}(\zeta, t) = \frac{1}{l(t)f(\zeta)} \cdot \frac{\partial \Psi}{\partial \zeta}(\zeta, t). \quad (4)$$

One of the advantages of the algorithm proposed in this article over the algorithm proposed in [1] is that the requirement of cavity area constancy is satisfied automatically:

$$\frac{dS_{cav}}{dt} = \oint_{\partial S} V ds = \oint_{\partial S} \frac{\partial \Psi}{\partial s} ds = 0. \quad (5)$$

Tangential and normal vectors can be derived from following equations:

$$\tau(\zeta, t) = \left(\frac{\partial x}{\partial s}, \frac{\partial y}{\partial s} \right) = \frac{1}{l(t)f(\zeta)} \left(\frac{\partial x}{\partial \zeta}, \frac{\partial y}{\partial \zeta} \right), \quad (6)$$

$$n(\zeta, t) = \left(\frac{\partial y}{\partial s}, -\frac{\partial x}{\partial s} \right) = \frac{1}{l(t)f(\zeta)} \left(\frac{\partial y}{\partial \zeta}, -\frac{\partial x}{\partial \zeta} \right). \quad (7)$$

Spatial coordinates x and y can be derived as follows:

$$\frac{\partial x}{\partial t}(\zeta, t) = \frac{1}{l(t)f(\zeta)} \left(U \frac{\partial x}{\partial \zeta} + V \frac{\partial y}{\partial \zeta} \right), \quad (8)$$

$$\frac{\partial y}{\partial t}(\zeta, t) = \frac{1}{l(t)f(\zeta)} \left(U \frac{\partial y}{\partial \zeta} - V \frac{\partial x}{\partial \zeta} \right), \quad (9)$$

where U is the tangential component of velocity and is chosen to keep the distribution of points along the cavity boundary unchanged.

Finally, we will use Bernoulli's equation:

$$\frac{\partial \Phi}{\partial t} \Big|_{x,y} + \frac{1}{2} \left(V^2 + \left(\frac{\partial \Phi}{\partial s} \right)^2 \right) + p = 0, \quad (10)$$

where $p(t)$ is an unknown pressure function. We can express the derivative of Φ with respect to t at constant ζ using the derivative of Φ with respect to t at constant x, y by the following formula:

$$\frac{\partial \Phi}{\partial t} \Big|_{\zeta} = \frac{\partial \Phi}{\partial t} \Big|_{x,y} + U \frac{\partial \Phi}{\partial s} + V^2. \quad (11)$$

Therefore, we have

$$\frac{\partial \Phi}{\partial t} \Big|_{\zeta} + p = \frac{1}{2} V^2 + \frac{U}{l(t)f(\zeta)} \frac{\partial \Phi}{\partial \zeta} - \frac{1}{2} \left(\frac{1}{l(t)f(\zeta)} \frac{\partial \Phi}{\partial \zeta} \right)^2. \quad (12)$$

Note that we need only partial derivatives of Φ with respect to spatial coordinates in equation (1), therefore we can replace Φ with the following function:

$$\tilde{\Phi}(\zeta, t) := \Phi(\zeta, t) + \int_0^t p(t') dt'. \quad (13)$$

The value of $p(t)$ still can be derived from equation (2) using values of $\tilde{\Phi}$ and V .

Approximation

For numerical computations, let us choose N points along the length of the cavity boundary. Each point corresponds to the fixed value $\zeta_i = \frac{i}{N}$, where $i = 1, 2, \dots, N$. Note that with this approach, tangential components of the points velocities and fluid velocities on the cavity boundary may be different, but their normal components are equal.

Cavity boundary parts with higher value of curvature require higher accuracy of calculations. Since the density of points distribution along the length of the boundary $f(\zeta)$ is an arbitrary function, we can choose it to increase number of points on cavity boundary parts of that kind.

To approximate equation (1), we should replace integral terms with quadrature formulas. Integral terms are represented by A and B operators. Operator A has a point of logarithmic singularity. Quadrature formulas for integrals of that kind has been proposed in paper [2]. The idea is to construct quadrature formulas that yield the exact result for the first $\frac{N}{2}$ partial sums of Fourier series. Since the error of approximation by Fourier series of infinitely differentiable periodic function decaying faster than $O(\frac{1}{N^n})$ for all $n > 0$, high order of approximation is granted by the quadrature formula. Operator B has also a point of singularity, but it is a removable singularity. To grant high order of approximation for this operator, it suffices to use rectangle quadrature formula. Its approximation error for the infinitely differentiable periodic function is also proved to decay faster than $O(\frac{1}{N^n})$ for all $n > 0$. These quadrature formulas are called quadrature formulas with no saturation. The term was coined by Babenko [3].

Most of the equations contain partial derivatives of Φ , Ψ , x and y with respect to spatial coordinates. A cubic spline for periodic functions is used to approximate them.

The distribution of points along the length of the cavity boundary is required to be kept unchanged. To satisfy it numerically, we require that the distances between adjacent points change in proportion to the length of the boundary. It can be expressed by the following system of equations:

$$\Delta s_i(t + \Delta t) = \frac{l(t + \Delta t) f\left(\frac{\zeta_{i-1} + \zeta_i}{2}\right)}{N}, \quad (14)$$

where $i = 1, 2, \dots, N$ and $\Delta s_i(t + \Delta t)$ is the distance between points with indices i and $i + 1$. Expressing $\Delta s_i(t + \Delta t)$ in terms of spatial coordinates x_i, y_i and velocities V_i, U_i at the moment of t , we derive a system of nonlinear equations for U_i . Iterative process is used to solve it.

To calculate potential and spatial coordinates functions on the next time step, the Runge-Kutta method is applied to equations (8), (9) and (12).

Algorithm features

Let us summarize all of the features that distinguish algorithm from the previous ones.

- Stream function Ψ is used instead of potential function Φ .
- Quadrature formulas with no saturation are used by the algorithm.
- A cubic spline for periodic functions is used to approximate partial derivatives with respect to spatial coordinates.
- The distances between adjacent points on the boundary change in proportion to the length of the boundary.
- Due to the choice of density function $f(\zeta)$, cavity boundary parts with higher value of curvature can be approximated using higher number of points.
- The Runge-Kutta method is used.

Main results

This section briefly describes the main results obtained by the implementation of the proposed algorithm.

First, let us consider the results for the problem stated in [1], i.e. evolution of a circular cylindrical cavity with a unit radius in a plane parallel flow. To approximate cavity boundary, 128 points were used. Near the right edge of the cavity a cumulative jet is formed. Since the forward end of the jet requires accurate computation, the following density function was used by the algorithm:

$$f(\zeta) = 1 - 0.99 \cos(2\pi\zeta), \quad (15)$$

i.e. the distribution is drastically denser on the forward end of the jet comparing it with the opposite side. The distribution is symmetrical about the x axis.

Calculation results are shown on Figure 1. Cavity boundaries for eight different moments of time are depicted, i.e. $t = 0, 0.17, 0.31, 0.43, 0.57, 0.71, 0.86$ and 0.93 . All of them except the last one have been chosen to demonstrate results in paper [1]. The proposed algorithm has shown similar results with those obtained in paper [1], but the cumulative jet

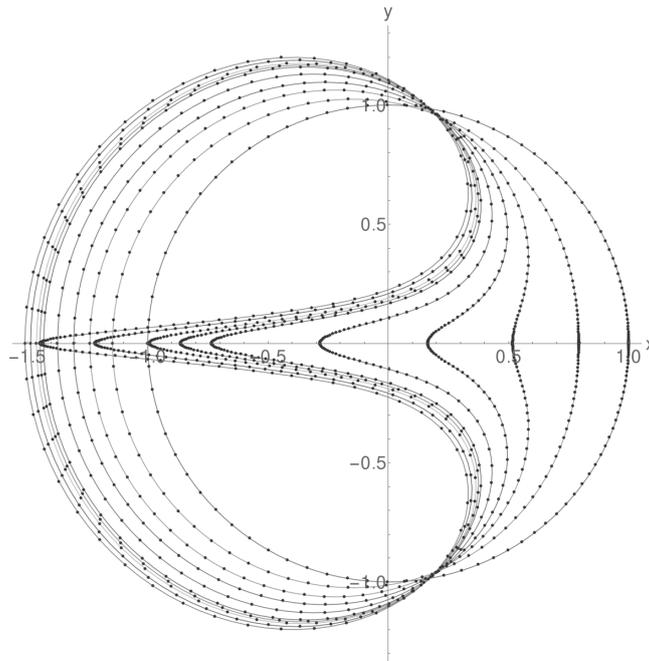


Figure 1: Circular cavity

calculated by the algorithm was closer to hitting the opposite side. The left-most point of the cumulative jet and the point on the opposite side velocities are shown on Figure 2. Note that cumulative jet velocity turns stable in course of time. The pressure function is shown on Figure 3.

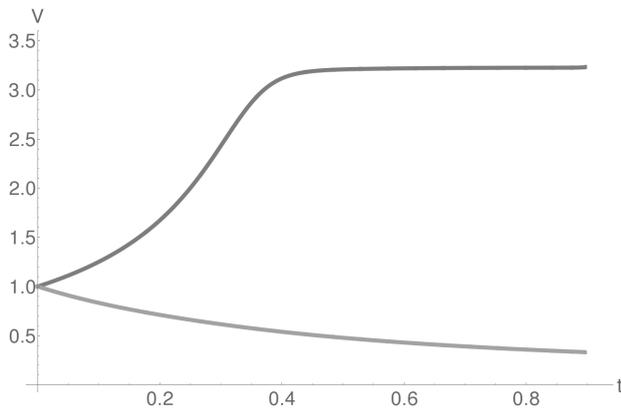


Figure 2: Cumulative jet and opposite side velocities

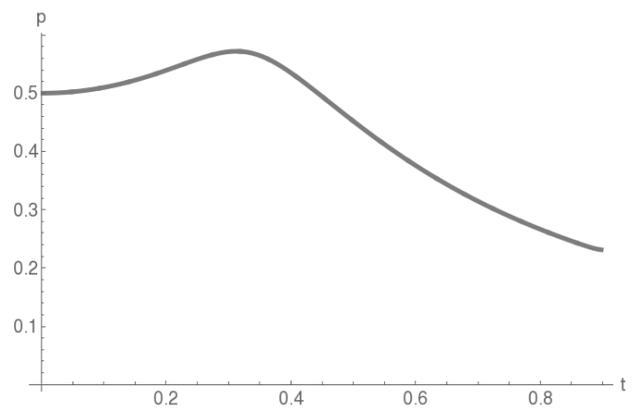


Figure 3: Pressure within cavity

To verify correctness of the proposed algorithm, conservation of energy, momentum and area were checked. The following equations were used to calculate energy and momentum:

$$E_c = -\frac{1}{2} \int_0^1 \Phi \frac{\partial \Psi}{\partial \zeta} d\zeta, \quad (16)$$

$$I_x = - \int_0^1 \Phi \frac{\partial y}{\partial \zeta} d\zeta. \quad (17)$$

When approximating the cavity boundary with 128 points, overall errors for them did not exceed $3 \cdot 10^{-4}$.

Let us consider other applications of the proposed algorithm. Figures 4 and 5 show calculation results for two different cases of flow past elliptic cavity. The former shows results for semi-major axis $a = 1.5$ and semi-minor axis $b = 1.0$ elliptic cavity on the assumption that normal components of boundary velocities are equal to zero in the initial time. In latter case normal components of velocities are not equal to zero at initial time and are given by the following formula:

$$V(\phi) = \frac{(b^3 - 2a^2b) \sin^2 \phi \cos \phi - a^2b \cos^3 \phi}{(b^2 \cos^2 \phi + a^2 \sin^2 \phi)^{\frac{5}{2}}}, \quad (18)$$

where ϕ is an angular coordinate of cavity boundary point. a and b are equal to 1.2 and 1.0, respectively. The latter case demonstrates that the proposed algorithm is suitable for calculation of very thin cumulative jets.

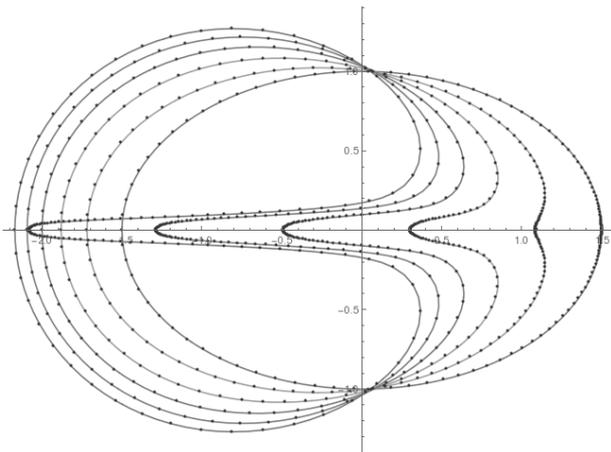


Figure 4: Elliptic cavity ($V = 0$)

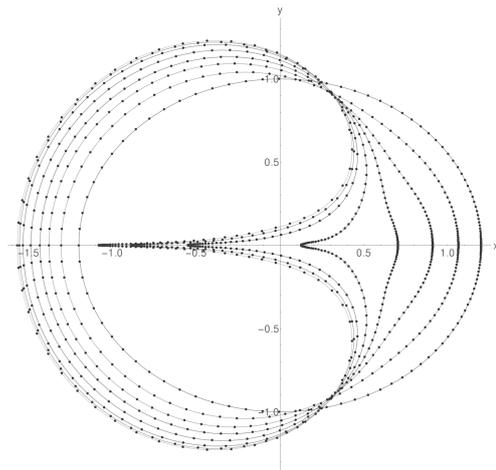


Figure 5: Elliptic cavity ($V \neq 0$)

Finally, the algorithm was applied to the problems of cavity rising under the forces of gravity and motion near a wall. Results for them are shown on Figures 6 and 7. Corresponding conservation laws were also checked to verify algorithm correctness in these cases. Note that potential energy should be taken into the account in the first case. The following equation is used to calculate it:

$$E_g = \frac{1}{2} \int_0^1 gy^2 \frac{\partial x}{\partial \zeta} d\zeta. \quad (19)$$

Results obtained for the problem of cavity rising are similar to the results obtained in paper [4].

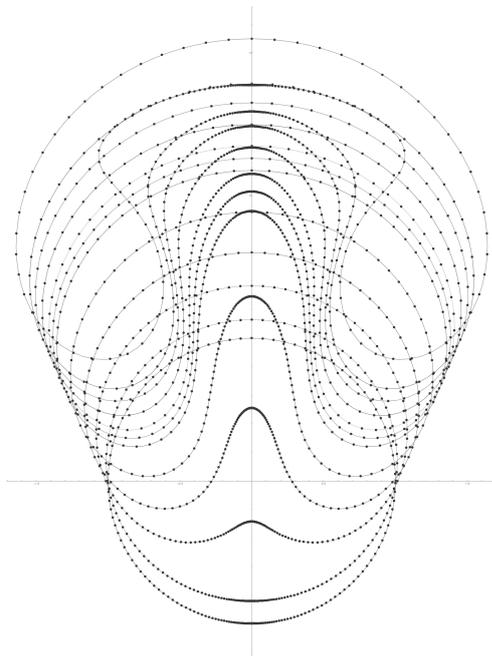


Figure 6: Cavity rising

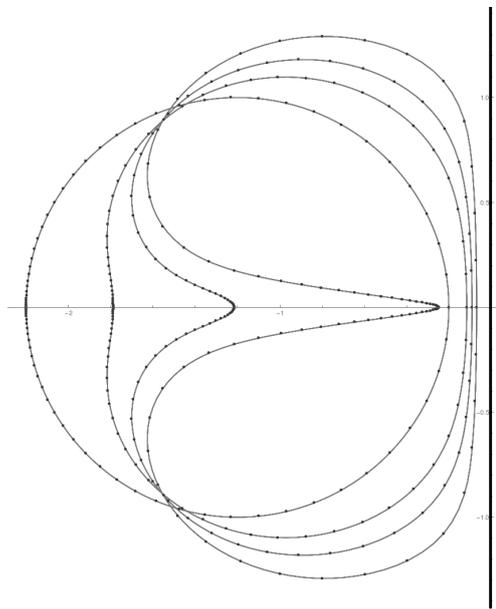


Figure 7: Cavity deformation near the wall

Acknowledgements

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