Calculus on Smith-Volterra-Cantor sets

Alireza Khalili Golmankhaneh*
* *Department of Physics, Urmia Branch,*
Islamic Azad University, Urmia, Iran

Summary. In this paper, we study the F^α-calculus triadic Cantor set which involves fractional local derivatives. The functions with fractal support are not differentiable and integrable in the sense of standard calculus. Triadic Cantor sets have self-similar properties and fractional dimensions which exceeds theirs the topological dimensions. We have generalized F^α-calculus on the triadic Cantor set to the Smith-Volterra-Cantor sets. We have suggested a calculus on the middle-ϵ, $(0<\epsilon<1)$ Cantor sets for the different value of the ϵ. Differential equations on Smith-Volterra-Cantor sets have been solved. Using the illustrative samples we present the results.

Basic tools in the fractal calculus

In this section, we summarize F^α-calculus without proofs [1]. We review the Cantor-Like sets and theirs properties as follows[6] :

Middle-ϵ Cantor Sets:
Let us consider the unit interval $I=[0,1]$ then in the first iteration remove the an open interval of length ϵ of $[0,1]$ from the middle of the I so we have

$$F_1^\epsilon=\left[0,\frac{1}{2}(1-\epsilon)\right]\cup\left[\frac{1}{2}(1+\epsilon),1\right]. \tag{1}$$

In the second iteration, we pick up two disjoint interval with length ϵ from the middle of each of the interval in the F_1^ϵ then arrive at

$$F_2^\epsilon=\left[0,\frac{1}{2}(1-\epsilon)^2\right]\cup\left[\frac{1}{4}(1-\epsilon)(1+\epsilon),\frac{1}{2}(1-\epsilon)\right]\cup$$
$$\left[\frac{1}{2}(1+\epsilon),\frac{1}{2}\left((1+\epsilon)+\frac{1}{2}(1-\epsilon)^2\right)\right]\cup\left[\frac{1}{4}(1-\epsilon)(1+\epsilon),\frac{1}{2}(1-\epsilon)\right]\cup \tag{2}$$
$$\left[\frac{1}{2}(1+\epsilon)\left(1+\frac{1}{2}(1-\epsilon)\right),1\right]. \tag{3}$$

Continuing iteration by picking up an open subinterval of length ϵ from the middle the disjoint intervals we lead ϵ-Cantor set as

$$F^\epsilon=\bigcap_{k=1}^{\infty}F_k^\epsilon \tag{4}$$

where F^ϵ has self-similarity property with the fractional dimension. The ϵ-Cantor sets has zero Lebesgue measure [6]. Namely,

$$\mathrm{L}_m(F^\epsilon)=\lim_{k\to\infty}\mathrm{L}_m(F_k^\epsilon)=\lim_{k\to\infty}(1-\epsilon)^k=0 \tag{5}$$

Hausdorff dimension of Middle-ϵ Cantor Sets:
For every Middle-ϵ Cantor sets the Hausdorff dimension is given by

$$\dim_H(F^\epsilon)=\frac{\log 2}{\log 2-\log(1-\epsilon)}. \tag{6}$$

where $H(F^\epsilon)$ is the Hausdorff measure which is used to derive Hausdorff dimension [6].
Remark 1. If we choose $\epsilon=1/3$, $\epsilon=1/4$, $\epsilon=1/5$, then we have the Cantor triadic set, 4-adic-type Cantor-like set, and 5-adic-type Cantor-like set/ middle 0.5-Cantor sets, respectively.

The mass function and the integral staircase
If F is a fractal set then it is the subset of $I=[a,b],\ a,\ b\in\Re$ (Real-line). The flag function for F is indicated by $\theta(F,I)$ and is defined [1],

$$\theta(F,I)=\begin{cases}1 & \text{if } \ F\cap I\neq\emptyset\\ 0 & \text{otherwise.}\end{cases} \tag{7}$$

The integral staircase function $S_F^\alpha(x)$ of order α for a fractal set F is defined in [1] by

$$S_F^\alpha(x)=\begin{cases}\gamma^\alpha(F,a_0,x) & \text{if} \quad x\geq a_0\\ -\gamma^\alpha(F,a_0,x) & \text{otherwise,}\end{cases} \tag{8}$$

where a_0 is an arbitrary real number.
A point x is a point of change of a function f if f is not constant over any open interval (a, d) involving x. The set **S**ch$f$ is called the points of change of f [1].
The γ-dimension of $F \cap [a, b]$ is

$$\begin{aligned} \dim_\gamma(F \cap [a,b]) &= \inf\{\alpha : \gamma^\alpha(F, a, b) = 0\} \\ &= \sup\{\alpha : \gamma^\alpha(F, a, b) = \infty\}. \end{aligned} \tag{9}$$

The integral staircase function $S_F^\alpha(x)$ of order α for a fractal set F is defined in [1] by

$$S_F^\alpha(x) = \begin{cases} \gamma^\alpha(F, a_0, x) & \text{if} \quad x \geq a_0 \\ -\gamma^\alpha(F, a_0, x) & \text{otherwise,} \end{cases} \tag{10}$$

where a_0 is an arbitrary real number.
A point x is a point of change of a function f if f is not constant over any open interval (a, d) involving x. The set **S**ch$f$ is called the points of change of f [1].
The γ-dimension of $F \cap [a, b]$ is

$$\begin{aligned} \dim_\gamma(F \cap [a,b]) &= \inf\{\alpha : \gamma^\alpha(F, a, b) = 0\} \\ &= \sup\{\alpha : \gamma^\alpha(F, a, b) = \infty\}. \end{aligned} \tag{11}$$

If **S**ch(S_F^α) is a closed set and every point in it is limit point, so that **S**ch(S_F^α) is called α-perfect. For the F-limit and the F- continuity definitions we refer the reader to [1].

F^α-Differentiation

If F is an α-perfect set then the F^α-derivative of f at x is [1]

$$D_F^\alpha f(x) = \begin{cases} \text{F}-\lim_{y \to x} \frac{f(y)-f(x)}{S_F^\alpha(y)-S_F^\alpha(x)}, & \text{if,} \quad x \in F, \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

if the limit exists.
In the next section, we will generalize the F^α-calculus on the Middle-ϵ Cantor Sets.

References

[1] A. Parvate, A. D. Gangal., Calculus on fractal subsets of real-line I: Formulation, *Fractals*, 17(01), 53–148, 2009.

[2] S. Satin, A. Parvate, A. D. Gangal, Fokker-Planck equation on fractal curves, *Chaos, Soliton Fract.*, 52 (2013), 30-35.

[3] A.K. Golmankhaneh, D. Baleanu, Fractal calculus involving Gauge function, *Commun. Nonlinear Sci.*, 37 (2016), 125-130.

[4] A.K. Golmankhaneh, D. Baleanu, New derivatives on the fractal subset of Real-line, *Entropy*, 18(2)(2016), 1-13.

[5] A.K. Golmankhaneh, D. Baleanu, Diffraction from fractal grating Cantor sets, *Journal of Modern Optics*, 63(14) (2016), 1364-1369.

[6] D. Robert, W. Urbina, On Cantor-like sets and Cantor-Lebesgue singular functions, arXiv preprint arXiv:1403.6554 (2014).