Decentralized guaranteed cost control for synchronization in networks of linear singularly perturbed systems*

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Summary. In this work we are providing results on decentralized guaranteed cost control design for synchronization of linear singularly perturbed systems connected by undirected links that are fixed in time. We show that we can proceed to a time-scale separation of the overall network dynamics and design the controls that synchronize the slow dynamics and the fast ones. This is done by transforming the problem of synchronization into a simultaneous stabilization one. Applying the joint control actions to the network of singularly perturbed systems we obtain an approximation of the synchronization behavior imposed for each scale. Moreover, the synchronization can be done with a guaranteed total energy cost.

Problem formulation

We consider a network of n identical singularly perturbed linear systems. For any $i = 1, \ldots, n$, the ith system at time t is characterized by the state $(x_i(t), z_i(t)) \in \mathbb{R}^{n_x + n_z}$ and there exists a small $\varepsilon > 0$ such that its dynamics is given by:

$$\begin{cases} \dot{x}_i(t) = A_{11}x_i(t) + A_{12}z_i(t) + B_1 u_i(t) \\ \varepsilon\dot{z}_i(t) = A_{21}x_i(t) + A_{22}z_i(t) + B_2 u_i(t), \quad i = 1, \ldots, n \end{cases} \tag{1}$$

where $u_i \in \mathbb{R}^m$ is the control input and $A_{11} \in \mathbb{R}^{n_x \times n_x}, A_{12} \in \mathbb{R}^{n_x \times n_z}, A_{21} \in \mathbb{R}^{n_z \times n_x} A_{22} \in \mathbb{R}^{n_z \times n_z}$ while $B_1 \in \mathbb{R}^{n_x \times m}, B_2 \in \mathbb{R}^{n_z \times m}$ such that $rank(B_1) = rank(B_2) = m$. We consider that the n systems are interconnected in a network described by a graph $\mathcal{G}$ which is a couple $(\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \{1, \ldots, n\}$ represents the vertex set and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the edge set. In the sequel we suppose that the graph is undirected meaning that $(i, j) \in \mathcal{E} \Leftrightarrow (j, i) \in \mathcal{E}$. We also assume that $\mathcal{G}$ has no self-loop (i.e. $\forall i = 1, \ldots, n$ one has $(i, i) \notin \mathcal{E}$). A weighted adjacency matrix associated with $\mathcal{G}$ is $G = [g_{i,j}] \in \mathbb{R}^{n \times n}$ such that $\begin{cases} g_{ij} > 0 \text{ if } (i, j) \in \mathcal{E} \\ g_{ij} = 0 \text{ otherwise} \end{cases}$. The corresponding Laplacian matrix is $L = [l_{ij}] \in \mathbb{R}^{n \times n}$ defined by $\begin{cases} l_{ii} = \sum_{j=1}^n g_{i,j}, \forall i = 1, \ldots, n \\ l_{ij} = -g_{i,j} \text{ if } i \neq j \end{cases}$. It is noteworthy that L is symmetric and if $\mathcal{G}$ is connected its eigenvalues satisfy

$$0 = \lambda_1 < \frac{4}{n(n-1)} \leq \lambda_2 \leq \ldots \leq \lambda_n < n.$$

Definition 1 *The n singularly perturbed systems defined by* (1) *achieve asymptotic synchronization using local information if there exists a state feedback controller of the form*

$$u_i(t) = K_1 \sum_{j=1}^n g_{i,j}(x_i(t) - x_j(t)) + K_2 \sum_{j=1}^n g_{i,j}(z_i(t) - z_j(t)), \quad K_1 \in \mathbb{R}^{m \times n_x}, K_2 \in \mathbb{R}^{m \times n_z} \tag{2}$$

such that

$$\lim_{t \to \infty} \|x_i(t) - x_j(t)\| = 0 \text{ and } \lim_{t \to \infty} \|z_i(t) - z_j(t)\| = 0.$$

The main goal of this paper is the characterization of the feedback controllers (2) that use local information and asymptotically synchronize the singularly perturbed systems defined by (1) with a guaranteed energy cost *i.e.*

$$\sum_{i=1}^n \int_0^\infty u_i(t) R u_i(t) \leq \bar{J} \tag{3}$$

where $R \in \mathbb{R}^{m \times m}$ is a positive definite matrix and $\bar{J}$ is the guaranteed cost that will be defined later.

Preliminaries on synchronization

In [1] we proposed a decentralized control design for the synchronization of systems (1). In order to do that we introduced the vector $x(t) = (x_1(t)^\top, \ldots, x_n(t)^\top)^\top$ and $z(t) = (z_1(t)^\top, \ldots, z_n(t)^\top)^\top$ collecting the slow and fast components of the individual states. We also introduced the orthonormal matrix $T \in \mathbb{R}^{n \times n}$ (i.e. $TT^\top = T^\top T = I_n$) such that

$$TLT^\top = D = diag(\lambda_1, \lambda_2, \ldots, \lambda_n).$$

With these notation we show that the synchronization problem of the n systems in (1) is equivalent with the state feedback simultaneous stabilization (SFSS) of the following $n - 1$ systems:

$$\begin{cases} \dot{\tilde{x}}_i(t) = (A_{11} - \lambda_i B_1 K_1)\tilde{x}_i(t) + (A_{12} - \lambda_i B_1 K_2)\tilde{z}_i(t) \\ \varepsilon \dot{\tilde{z}}_i(t) = (A_{21} - \lambda_i B_2 K_1)\tilde{x}_i(t) + (A_{22} - \lambda_i B_2 K_2)\tilde{z}_i(t), \end{cases} \quad i = 2, \ldots, n \tag{4}$$

where $\tilde{x}(t) = (\tilde{x}_1(t), \ldots \tilde{x}_n(t))^\top = (T \otimes I_{n_x})x(t)$ and $\tilde{z}(t) = (\tilde{z}_1(t), \ldots \tilde{z}_n(t))^\top = (T \otimes I_{n_z})z(t)$. Moreover, in [1] we showed that SFSS of systems (4) can be achieved in a decentralized manner provided that the pairs (A_{22}, B_2) and (A_0, B_0) are controllable (where $A_0 = A_{11} - A_{12}A_{22}^{-1}A_{21}$, $B_0 = B_1 - A_{12}A_{22}^{-1}B_2$).

Since the synchronization problem of systems (1) is reformulated as the simultaneous stabilization problem of (4), the synchronization performances are also translated to stability ones. This justify the introduction of the following individual quadratic costs:

$$\widetilde{\mathcal{J}}_i = \int_0^\infty \left(\widetilde{\mathbf{x}}_i^\top(t)Q\widetilde{\mathbf{x}}_i(t) + \widetilde{u}_i^\top(t)R\widetilde{u}_i(t)\right)dt, \quad i = 2, \ldots, n \tag{5}$$

where $\widetilde{\mathbf{x}}_i(t) = \left[\tilde{x}_i^\top(t), \tilde{z}_i^\top(t)\right]^\top$, $Q = Q^\top > 0$, $R = R^\top > 0$ and

$$\widetilde{u}_i(t) = -\lambda_i K_1 \tilde{x}_i(t) - \lambda_i K_2 \tilde{z}_i(t), \; \forall i \in 1, \ldots, n \tag{6}$$

Main results

In this section we show that minimizing an upper-bound for the costs $\widetilde{\mathcal{J}}_i$ leads to a guaranteed cost $\bar{J}$ in (3).

Proposition 2 *If there exists a guaranteed cost $\beta_i > 0$ such that the closed-loop value of the cost function (5) satisfies $\widetilde{\mathcal{J}}_i \leq \beta_i$ for all $i = 2, \ldots, n$ then a guaranteed cost of value $(n - 1) \max\limits_{i=2,\ldots,n} (\beta_i)$ is ensured for the global control performance required to asymptotically synchronize the collective closed loop dynamics (1).*

The proof is based on the fact that

$$\sum_{i=1}^n \int_0^\infty u_i^\top(t)Ru_i(t) = \sum_{i=2}^n \int_0^\infty \widetilde{u}_i^\top(t)R\widetilde{u}_i(t) \tag{7}$$

Based on the results in [2, 3] we will remove the dependence of β_i on the eigenvalues λ_i of the Laplacian matrix L. First we note that (4) can be written as

$$\dot{\widetilde{\mathbf{x}}}_i(t) = A_\varepsilon \widetilde{\mathbf{x}}_i(t) + D_\varepsilon \widetilde{u}_i(t), \quad \forall i = 2, \ldots, n \tag{8}$$

where

$$A_\varepsilon = \begin{pmatrix} A_{11} & A_{12} \\ \varepsilon^{-1}A_{21} & \varepsilon^{-1}A_{22} \end{pmatrix}, \quad D_\varepsilon = \begin{pmatrix} B_1 \\ \varepsilon^{-1}B_2 \end{pmatrix}$$

and the control law is of the form

$$\widetilde{u}_i(t) = -F_i \, \mathcal{K} \, \widetilde{\mathbf{x}}_i(t) \tag{9}$$

where $\mathcal{K} = [K_1, \; K_2]$, and F_i denotes the uncertainty matrix such that $F_i = \lambda_i I_{n_x + n_z}$.

Assumption 1 *There exists ε^* such that for all $\varepsilon \in (0, \varepsilon^*]$, the pair $(A_\varepsilon, D_\varepsilon)$ is stabilizable.*

Then, we have the following result :

Theorem 3 *Consider the uncertain system (8) suppose the graph $\mathcal{G}$ is connected and Assumption 1 holds. Then, there exists $\varepsilon^* > 0$ such that for each $\varepsilon \in (0, \varepsilon^*]$ and for each $i = 2, \ldots, n$, the following Riccati equation :*

$$0 = P_\varepsilon A_\varepsilon + A_\varepsilon^\top P_\varepsilon - (\lambda^*)^2 P_\varepsilon D_\varepsilon \widetilde{R}^{-1} D_\varepsilon^\top P_\varepsilon + \widetilde{Q} \tag{10}$$

admits a positive definite symmetric solution P_ε. Moreover the controller $\tilde{u}_i(t) = -\lambda^* \, \mathcal{K} \, \widetilde{\mathbf{x}}_i(t), \quad \forall i \in 2, \ldots, n$ with $\lambda^* = \frac{4}{n(n-1)}$ and $\mathcal{K} = [K_1, \; K_2] = \widetilde{R}^{-1} D_\varepsilon^\top P_\varepsilon$ stabilizes (8). Moreover, for any given $\kappa > 0$, there exists a matrix $\widetilde{P}_\varepsilon$ such that $P_\varepsilon < \widetilde{P}_\varepsilon < P_\varepsilon + \kappa I_{n_x + n_z}$ defining the upper-bound $\beta_i = \widetilde{\mathbf{x}}_i(0)^\top \widetilde{P}_\varepsilon \widetilde{\mathbf{x}}_i(0)$ of the guaranteed cost associated with the controller $\tilde{u}_i(t)$ (i.e. $\widetilde{\mathcal{J}}_i \leq \beta_i$).

References

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