A Discrete Predator-Prey Conflict Model with Defense Term

Markus H. E. Messer*, Joachim A. Messer†
*Fachbereich Maschinenbau, Mechatronik, Materialtechnologie (Mechanical Engineering), Technische Hochschule Mittelhessen (University of Applied Sciences), Friedberg, Germany
†Institut für Theoretische Physik der Justus-Liebig-Universität, Gießen, Germany

Summary. In this paper, a logarithmic term (i.e.: defense term) is used for the first time to model defense actions of the prey in a discrete predator-prey system. The results show, that an additional dissipative defense term disturbs the cyclic behavior (i.e.: periodic solution) and leads to interesting collective phenomena. Increasing the value of the defense parameter results in period-doubling bifurcation followed by an onset of chaos in the discrete dynamical system.

Introduction

Predator-prey models can be found in many textbooks, such as [1]. A famous predator-prey model is the so-called Lotka-Volterra model, which was formulated independently by Lotka [2] and Volterra [3] in order to represent the dynamics of biological systems. The Lotka-Volterra model consists of a pair of first-order nonlinear differential equations, that have periodic solutions. This continuous model originally describes two interacting species, where one species stands for prey (e.g.: rabbits), and the other species represents some predator (e.g.: foxes).
Leslie discretized the continuous prey-predator system and numerically analyzed the discrete dynamical system [4-6]. Leslie's discrete model qualitatively conserves the characteristics of the original continuous model.
Seno considers an extension of Leslie's idea and presents a new discrete Lotka-Volterra predator-prey model in [7]. Furthermore, Seno shows in [7] that this discrete system is, independent of the time step size, dynamically consistent with the continuous system. In this article, we refer to Seno's dynamical system as the unperturbed predator-prey system.
In [8], a continuous predator-prey system is used to model the dynamics of a forest ecosystem, and Waldsterben (forest dieback) due to chemical pollution is simulated. For this purpose, a biostress term is derived in [8] from Lindblad's dissipative dynamics [9] leading to stressed Lotka-Volterra interactions (perturbed predator-prey system).
In [10], a hygienic stress term is obtained in order to model the spread of avian influenza using a discrete system, and numerical results of the difference equation are compared with measurement data. As in [8], the derivation of the hygienic stress term in [10] is also based on Lindblad's work [9]. The hygienic stress diminishes the virus's replication rate in host cells and is described by a logarithmic term in [10].

The dissipative predator-prey system

In this paper, a logarithmic term ($-g\,h\,P_k \ln(b\,P_k/r)$, see below) is used for the first time to model defense actions of the prey in a discrete predator-prey system. The resulting equations of our time-discrete model are:

$$H_{k+1} = e^{rh}\, H_k \left[1 - \Pi_h(P_k) \right];$$

$$P_{k+1} = e^{-\delta h} \left\{ P_k + c\,\frac{\Phi_P(h)}{\Phi_H(h)}\, e^{rh}\, H_k\, \Pi_h(P_k) \right\} - g\,h\,P_k \ln\!\left(\frac{b\,P_k}{r} \right),$$

with

$$\Pi_h(P_k) = \frac{\Phi_H(h)\, b\, P_k}{1 + \Phi_H(h)\, b\, P_k}, \quad \Phi_H(h) = \frac{e^{rh} - 1}{r}, \quad \Phi_P(h) = \frac{e^{\delta h} - 1}{\delta}.$$

We refer to this discrete dynamical system as the dissipative (or perturbed) predator-prey system, where H and P represent population sizes of prey and predator, respectively. h is the time step size, and g is the defense parameter. For $g = 0$ our defense term is zero, and our discrete predator-prey model is equal to Seno's model presented in [7] with the following parameters: $r = 1.0$ (intrinsic (malthusian) growth rate of prey), $\delta = 0.1$ (natural death rate of predator), $b = 1.0$ (predation coefficient), $c = 0.01$ (energy conversion rate from the predation to the predator's reproduction).
The equations shown above could be used to simulate the impact of a virus onto the human population. In this case, the virus is a predator to prey role of humans, if the induced disease is lethal. This is true for Swine Influenza A/H1N1 with a short incubation time. In this article, we do not limit the application to dynamics of biological systems, but our model should also allow the analysis of other systems. This means that the discrete predator-prey

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system suggested in this paper could for example be used in peace research to model a conflict between two parties, where the prey develops a defense action. The stress term in biology represents now in peace research a defense term. This might model the north-south conflict or the behavior of the first and second world with respect to the third or fourth world. Also in some cases it could model certain aspects of conflict scenarios. Hopefully, this helps to prevent conflicts or improves future conflict management.

Results

The unperturbed predator-prey system displays a regular cyclic behavior (periodic orbits) as shown by Seno [7]. Our results show, that an additional dissipative defense term disturbs this cyclic behavior and leads to interesting collective phenomena. The behavior of a completely cyclic discrete system can be changed drastically by this perturbation, and new qualitative features appear, when the perturbation becomes larger. Increasing the value of the defense parameter results in period-doubling bifurcation followed by an onset of chaos in our dissipative (or perturbed) predator-prey system.

Of course the model is very crude, because it is homogeneous, therefore considering the earth as a Petri dish. Obviously, this is in some sense unrealistic, because victim probabilities as well as defense actions will depend on inhomogeneities. But as a mathematical model it might function as a playground for testing structural and mathematical ideas.

The numerical solutions of the discrete iterations show that precisely at $gh = 2.0$ a phase transition occurs. At this point the type of the solution changes. One calculation is shown with $gh = 0.01$ (see figures 1 and 2). Figure 1 shows a trajectory in the phase plane (H, P) for $gh = 0.01$. The corresponding time series can be seen in figure 2. Increasing gh changes the system's behavior. For example, at $gh = 2.01$ we can see a completely other type of solution. We have calculated a sequence of steps from $gh = 0.0$ until $gh = 2.58$ having emphasis on a small variation of g if gh is close to 2.0. The value of h is in this sequence always 0.25. The phase transition is also very impressively seen at the orbit diagram (figure 3).

Another sequence of calculations for step width $h = 0.5$ again showed the phase transition at $gh = 2.0$. This implies that the effect is only seen in the discrete system. In the continuum limit, where $h \to 0$, g would need to reach infinity, to get the same effect.

In our dissipative (or perturbed) predator-prey model, oscillations occur for small gh ($gh < 2.0$). For increasing gh, a phase transition occurs at exactly $gh = 2.0$. And for large gh ($gh > 2.0$) there is period-doubling bifurcation followed by an onset of chaos. In our interpretation the predator is not beaten, but it survives in a chaotic manner.

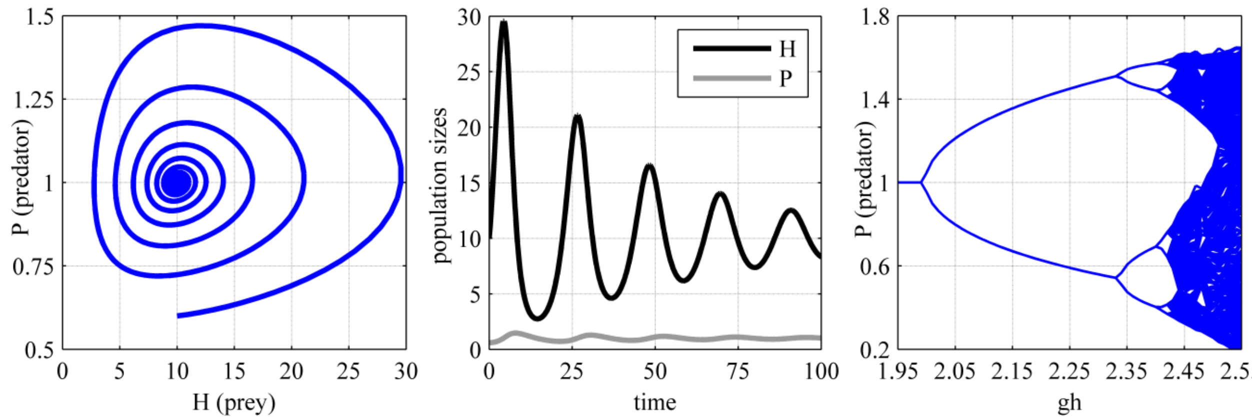

figure 1 (left): trajectory in the phase plane (H, P) for $gh = 0.01$, figure 2 (middle): time series of H and P for $gh = 0.01$, figure 3 (right): period-doubling bifurcation and onset of chaos in the orbit diagram; initial state: $H(0) = H_0 = 10.0$ and $P(0) = P_0 = 0.6$

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