Generalized Fractional Order Reset Element (GFrORE)

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Summary. This paper presents and studies the novel Generalized Fractional Order Reset Element (GFrORE). Fractional order filter realizations which enable control over the Q-factor of the filter is studied. Subsequently, the concept of reset in control system is applied to fractional order filters with arbitrary reset matrices, resulting in additional reset parameter γ along with fractional order α, which can be used to optimize the filter design. A general proof of stability is provided and sinusoidal input describing function method is used to analyse the effect of the generalized reset especially on its ability to reduce phase lag. Reset action is also found to affect corner frequency of the filters for negative values of γ and a rule of thumb is presented to design the filters by taking this into consideration. The Q-factor of the designed filter is also affected by reset action and this is used as an advantage in the presented numerical example for its use in the field of motion control applications.

Introduction

Fractional Order Calculus (FOC) has constantly been gaining prominence with active research in diverse applications. In the field of controls, design of controllers using FOC in comparison with classical integer-order controllers provides additional capability and flexibility in tuning and is seen favourably by industry. This has seen successful practical implementations presented in literature with utilization in cruise control by HosseinNia et al. [1], voltage regulation by Zamani et al. [2], control of time-delay systems by Hamamci [3], and fractional order systems by Zhao et al. in [4] and Ying et al. in [5]. Several works specifically dealing with the design, tuning and optimization of fractional order PID further shows its growing importance [6, 7, 8]. Despite its advantage, controllers designed using FOC belong to the family of linear controllers and hence suffer from fundamental limitations in linear control like water-bed effect, mid-frequency disturbance rejection etc.

Reset control is a non-linear technique which was first initiated by Clegg for a simple integrator [9]. This technique resets the state/s of the controller to zero when some pre-defined conditions are met, with the most commonly used condition being when the error hits zero. This reset concept has been extended by Horowitz and Rosenbaum to First Order Reset Element (FORE) [10] and by Hazeleger et al. to Second Order Reset Element (SORE) [11], where the reset action dynamics are presented from the linear control system perspective. The advantage of reset is seen in frequency domain as reduced phase lag without any effect in magnitude. Generalized FORE (GFORE) introduced by Guo et al. [12] further provides an additional parameter γ for greater flexibility in obtaining the required phase curve. These have been used advantageously to overcome the limitations of linear controllers mentioned earlier in several works ranging from process control to electrical systems, and motion control [13, 14, 15, 16, 17, 18].

It should come as no surprise that the advantages of FOC and reset have been combined with the fundamentals of fractional order Clegg integrator by Monje et al. [19], Valério et al. [20], HosseinNia et al. [21, 22, 23]. In this work, we present the novel **G**eneralized **Fr**actional **O**rder **R**eset **E**lement (GFrORE). Realization of fractional order filters and the advantages of one over the other are presented in the next section. The main contribution GFrORE is presented and analysed using describing function analysis in the subsequent section. The effect of having two additional parameters in comparison to an integer order linear filter, α to determine the fractional order and γ for the resetting matrix is studied. A numerical example from the field of motion control is provided to highlight the advantage of using GFrORE in practice followed by conclusions.

Fractional order filters

First and second order filters are the most common filters studied in literature and utilized in controls. Hence we focus mainly on fractional order filters with order in the range [0.5, 2.5], where the first and second order filters can be considered as candidates for representation of fractional order filters as shown below.

$$H^{\alpha}(s) = \frac{{\omega_0}^{\alpha}}{s^{\alpha} + {\omega_0}^{\alpha}} \quad \forall\ \alpha \in [0.5, 2.5]$$

$$H^{2\alpha}(s) = \frac{({\omega_0}^2)^{\alpha}}{(s^2)^{\alpha} + 2\beta_r(\omega_0 s)^{\alpha} + ({\omega_0}^2)^{\alpha}} \quad \forall\ 2\alpha \in [0.5, 2.5]$$

where ω_0 is the corner frequency and β_r is the damping coefficient.

However, the use of $H^{\alpha}(s)$ does not provide control of the filter Q-factor to the designer and worse results in an unstable filter for order greater than 2 ($\alpha > 2$). Hence, the realization using $H^{2\alpha}(s)$ is studied and used for realization of GFrORE later. This results in a stable realization with the additional advantage where the required Q-factor can be achieved using the formula provided below.

$$\beta_r = \frac{1}{2Q} - cos\Big(\frac{\alpha\pi}{2}\Big) \tag{1}$$

This realization can be used to realize even first order filters (instead of the conventional way), to get the required Q as shown in Fig. 1. While values of β_r greater than $-cos\left(\frac{\alpha\pi}{2}\right)$ result in constantly decreasing values of Q-factor, smaller values result in unstable realizations of the fractional order filters.

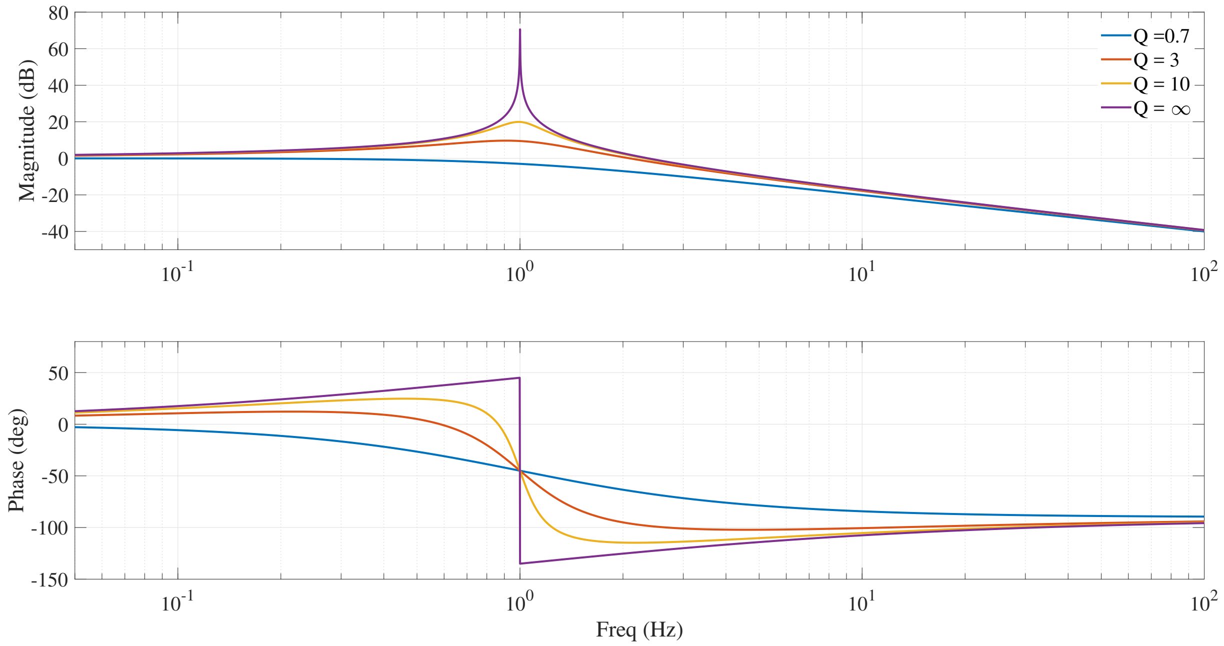

Figure 1: Bode plots of first order filter realized using $H^{2\alpha}(s)$ for different Q-factors

Generalized Fractional Order Reset Element

Generalized Reset Element

An integer order reset element is defined as

$$G_{RE} = \begin{cases} \dot{x}_r(t) = A_r x_r(t) + B_r e(t), & \forall e(t) \neq 0 \\ x_r(t^+) = A_\rho x_r(t), & \forall e(t) = 0 \\ u_r(t) = C_r x_r(t) + D_r e(t) \end{cases}$$

where A_r, B_r, C_r and D_r are the state matrices. The second equation is called *reset law* and A_ρ is the *resetting matrix* for the states and is defined as in [12] as

$$A_\rho = \gamma I$$

with I being the corresponding identity matrix and $\gamma \in \mathbb{R}$. Further $\gamma = 1$ results in the linear filter with no reset.
The stability criteria for a generic reset system is provided in [12] and it is found that the system has a globally asymptotically stable $2\pi/\omega$-periodic solution under sinusoid input with arbitrary frequency $\omega > 0$ iff

$$|\lambda(A_\rho e^{\frac{\pi}{\omega}A})| < 1 \tag{2}$$

Sinusoidal input describing function analysis can be used to obtain frequency behaviour of the filter as in [12] as

$$G(j\omega) = {C_r}^T(j\omega - A_r)^{-1}(I + j\Theta_\rho(\omega))B_r \tag{3}$$

where,

$$\Theta_\rho(\omega) = \frac{2}{\pi}\frac{e^{\frac{\pi A_r}{\omega}} + I}{A_\rho e^{\frac{\pi A_r}{\omega}} + I}\frac{I - A_\rho}{\frac{{A_r}^2}{\omega^2} + I}$$

The advantage of reset action is in reduced phase lag compared to its linear counterpart. However, it is seen from Eqn. 3 that this is achieved only when $\Theta_\rho(\omega) > 0$ which adds an additional constraint on A_ρ as

$$e^{-\frac{\pi}{\omega}A} < A_\rho < I \tag{4}$$

The stability criteria of Eqn. 2 and reduced phase lag requirement which leads to Eqn. 4 are satisfied for all frequencies only when $|\gamma| \leq 1$. It must be noted that while Eqn. 2 can be used to analyse the overall stability of a reset system, describing function based frequency response can be also be used for the same in open loop by using Gain Margin (GM), Phase Margin (PM) and Modulus Margin (MM - inverse of the infinity norm of the sensitivity function) as indicators of stability.

GFrORE

Practical realization of GFrORE requires the approximation of fractional order filter $H^{2\alpha}(s)$. CRONE approximation is used in this work [24]. This CRONE approximated transfer function is converted into state space representation, upon which reset can be applied. The stability criteria submitted for the generalized reset element is applicable for the approximated GFrORE. State space representation is also used to obtain the describing function based frequency response using Eqn. 3. These are studied for values of 2α (order of filter) in range $[0.5, 2.5]$, while γ is varied in range $[-1, 1]$. The frequency responses obtained for a 1.5^{th} order filter for different values of γ are shown in Fig. 2. A study of the frequency responses reveals 3 interesting properties which are dealt with below.

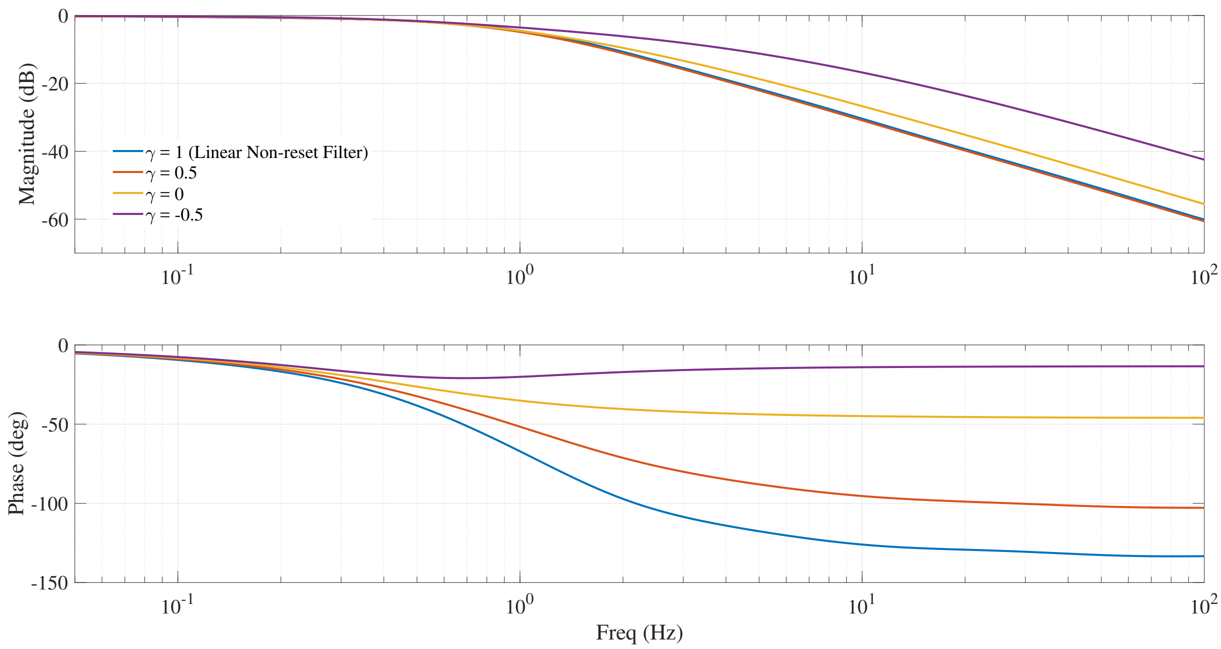

Figure 2: Bode plots of GFrORE obtained using describing function for order $2\alpha = 1.5$ for different values of γ

Reduced phase lag

Fig. 2 clearly shows the significant reduction in phase lag of GFrORE compared to its linear counterpart ($\gamma = 1$) with the amount of reduction related to γ which is the level of non-linearity. In the case of GFORE presented and studied in [12], the relation between phase lag decrease and γ is close to linear. However, an analysis of GFrORE shows a non-linear relation between phase lag and γ, with non-linearity increasing as the order increases. This is shown in Fig. 3. It is also noticed that the phase lag converges for all values of α when the resetting factor $\gamma \leq -0.2$. However, this change in phase lag for different values of γ does not affect the magnitude slope as seen in Fig. 2. Its effect on the corner frequency is dealt with later. This property of GFrORE highlights the advantage of applying generalized reset to fractional order filters, since α can be chosen to obtained the required magnitude slope, while the value of γ can be chosen to obtain the necessary phase lag.

Change in corner frequency

An interesting property seen with GFrORE is in the shifting of corner frequency for values of $\gamma \leq 0$. While the magnitude slope is determined by 2α, it is noticed that the corner frequency shifts away from ω_0 to $\omega_0' > \omega_0$, with the value of ω_0' moving exponentially away from ω_0 with a decreasing value of γ. This is shown in Fig. 4 for $\omega_0 = 1$. This increase is extremely similar in value for all values of α considered and can be linearised to get the approximate relation (rule of thumb) as:

$$\omega_0' = \begin{cases} \omega_0, & for \;\; \gamma > 0 \\ 10^{\frac{-\gamma}{0.8}}\omega_0, & for \;\; -0.8 \leq \gamma \leq 0 \\ 10^{\frac{-\gamma-0.625}{0.175}}\omega_0, & for \;\; -0.975 \leq \gamma \leq -0.8 \end{cases}$$

This property of shifted corner frequency is not reflected in the phase plot. This rule of thumb can be used for the design of GFrORE, by determining the value of ω_0 for the corner frequency ω_0' where the filtering action is required to start.

Effect on Q-factor

It is also noticed that reset affects the Q-factor by significantly reducing the resonance peak. So the Q-factor of GFrORE (say Q_r) is not the same as that of the designed fractional order filter. As a result of this large reduction, the magnitude of resonance peak does not vary significantly for different values of β_r. However, the phase behaviour is affected and this is

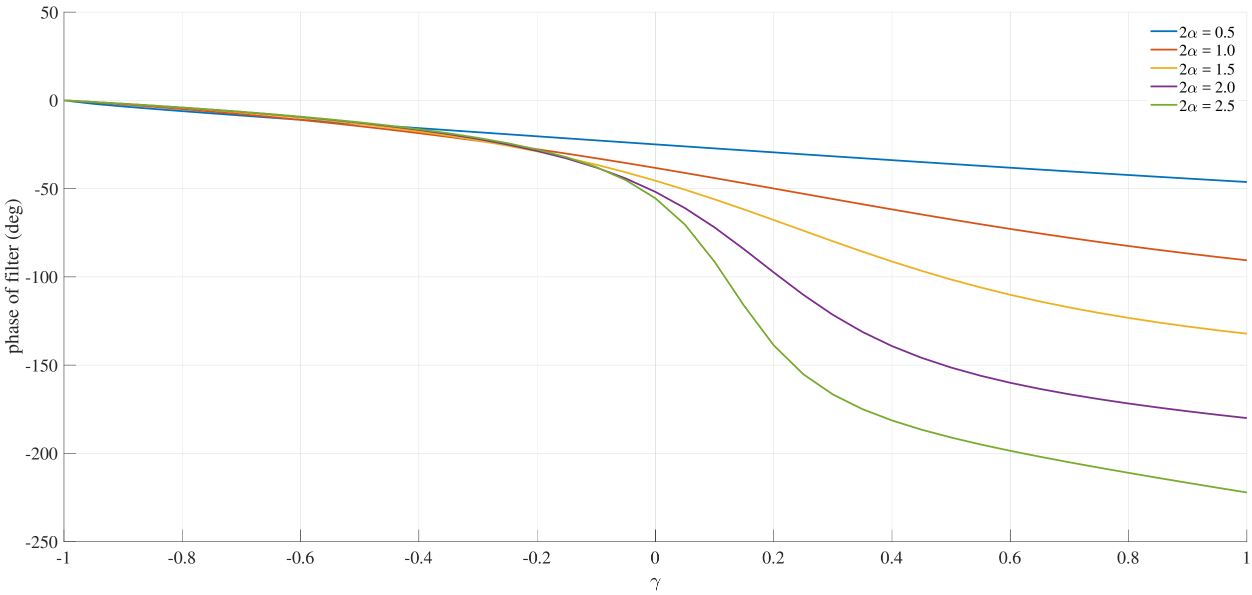

Figure 3: Variation of phase lag in degrees vs γ for different values of α

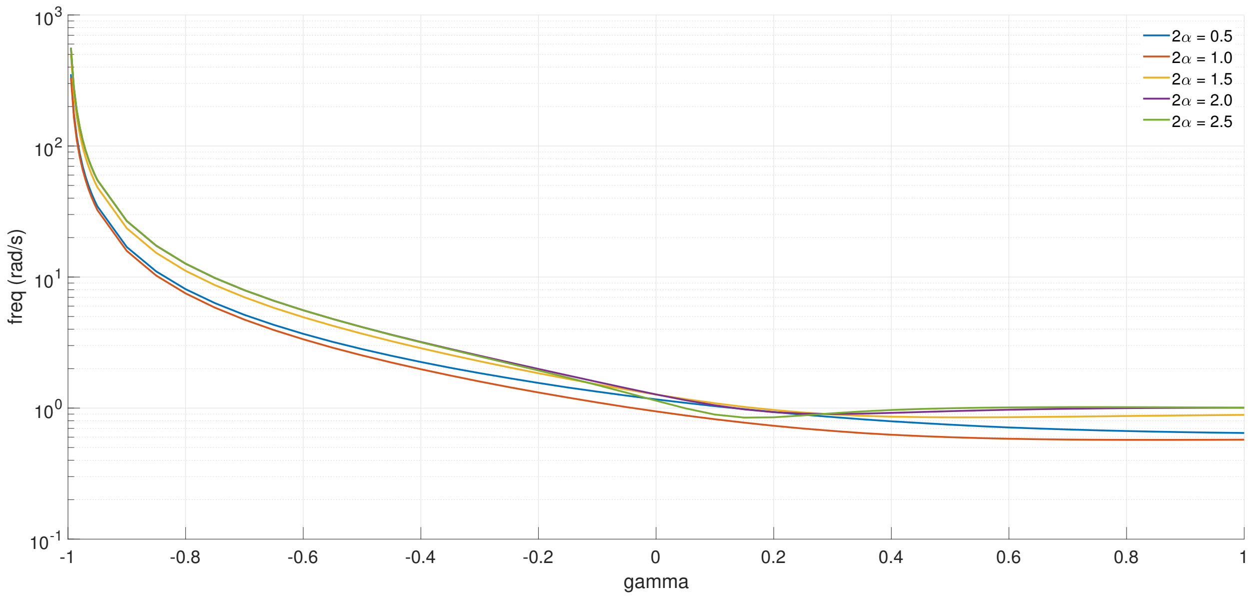

Figure 4: Shift of corner frequency with change in γ for different values of α

shown in Fig. 5. This property of reset can be used to design GFrORE where the value of β_r is solely used to shape the phase behaviour while any change seen in magnitude behaviour is insignificant for most practical purposes. A ripple like behaviour is also noticed in the phase plots for smaller values of β_r. This could be due to the approximation technique used and needs further study.

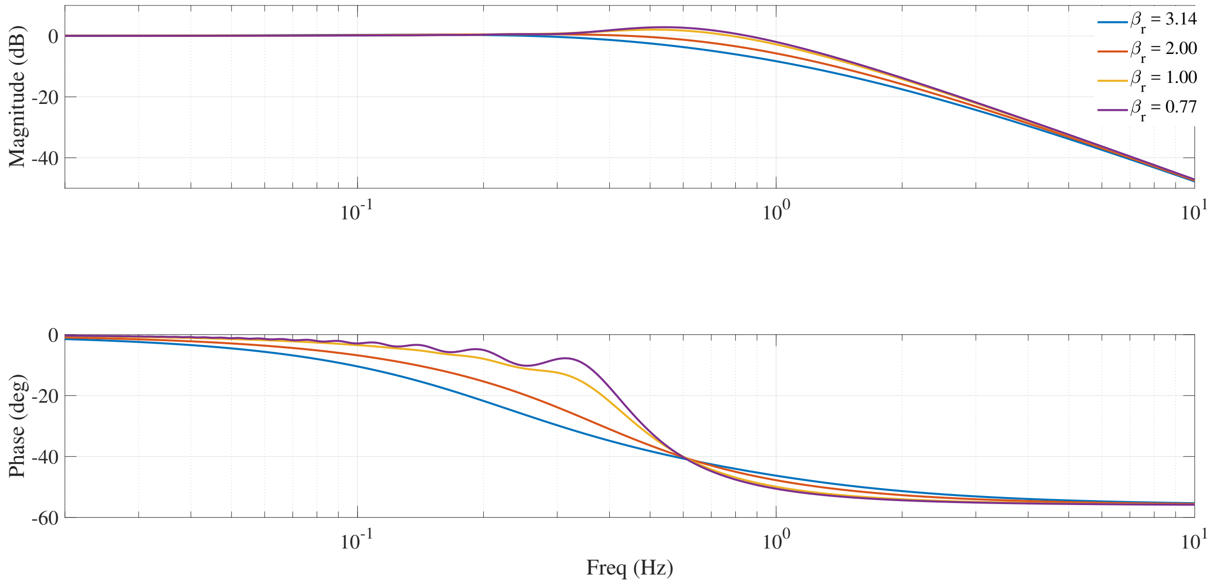

Figure 5: Frequency Response of GFrORE of order $2\alpha = 2.5$ with $\gamma = 0$ for different values of β_r

The reduction of phase lag and Q-factor are used in the numerical example presented to highlight the ability of GFrORE to overcome some of the limitations of linear control.

Numerical Example

Consider a 4^{th} order motion control system similar to a non-collocated double mass single spring system whose transfer function is given below. Industrial standard PID designed using rules of thumb will be considered for control for easy comparison of performance of GFrORE.

$$P(s) = \frac{{w_p}^2}{s^2\left(s^2 + \frac{2sw_p}{Q_p} + {w_p}^2\right)}$$

where w_p is the resonance frequency of $100Hz$ and Q_p is the Q-factor = 30.
The system phase sharply goes to $-360\,\mathrm{deg}$ at resonance limiting the achievable bandwidth to below w_p. Since the resonance peak is quite large, achieving bandwidths close to w_p is also impossible with PID, since the resonance peak results in additional cross-over frequencies. This is solved in the industry through a low pass filter whose main function is to introduce sufficient phase lag around w_p rather than attenuating the resonance peak. It additionally serves the purpose of attenuating noise at higher frequencies.
To highlight the advantage of using fractional order and then reset, controllers are designed using first order, second order, fractional order filters and GFrORE. The system specifications and parameters of the filters are provided below.

- System Specifications
  - Achieve highest possible bandwidth
  - Min PM = $30\,\mathrm{deg}$
  - Min MM = $6dB$
- Since the system is Type 2, PD is used instead of PID and is designed using rules of thumb. The transfer function of PD designed using such rules of thumb for some bandwidth ω_c is provided below.

$$PD(s) = K\frac{1 + \frac{3s}{\omega_c}}{1 + \frac{s}{3\omega_c}}$$

  where K is adjusted to obtain the exact bandwidth ω_c. The parameters of the different filter configurations used and the bandwidth, PM and MM achieved are provided below.

- First order filter
  - Corner frequency $\omega_f = 80Hz$
- Second order filter
  - Corner frequency $\omega_f = 110Hz$ with $Q = 1$
- Fractional order filter
  - $2\alpha = 2.5$
  - Corner frequency $\omega_f = 140Hz$ with $Q = 0.97$
- GFrORE
  - $2\alpha = 2.5, \gamma = 0$
  - Corner frequency $\omega_f = 170Hz$ with $Q = 4300$ (This is the Q factor of the fractional order filter. As noted before with reset, the Q resonance peak is attenuated significantly.)

The bandwidth ω_c achieved in first three cases is 24 Hz. A marginal increase to 25 Hz is achieved with GFrORE. The attenuation of original Q-factor through reset allows for greater freedom in shaping the phase of filter and is used to achieve the slight improvement. This can be seen in the large value of Q compared to the other 2 cases. Open loop bode plots are provided in Fig. 6 for all cases along with step responses in Fig. 7. Higher attenuation is achieved at higher frequencies for higher order filters in the case of both linear and GFrORE as seen in the open loop plots, which in turn result in comparatively faster settling.
The main advantage of GFrORE however over it's non-reset linear fractional counterpart is seen in the sensitivity bode plot provided in Fig. 8. While increase in the filter order and the use of fractional order ensures more noise attenuation and faster settling time, the water-bed effect results in higher sensitivity at lower frequencies for larger orders of LPF in the linear case. However, in the case of GFrORE, although the peak of sensitivity is still limited to $6dB$ by the system specifications and the order is $2\alpha = 2.5$, the system is least sensitive among the cases considered as seen in Fig. 8.

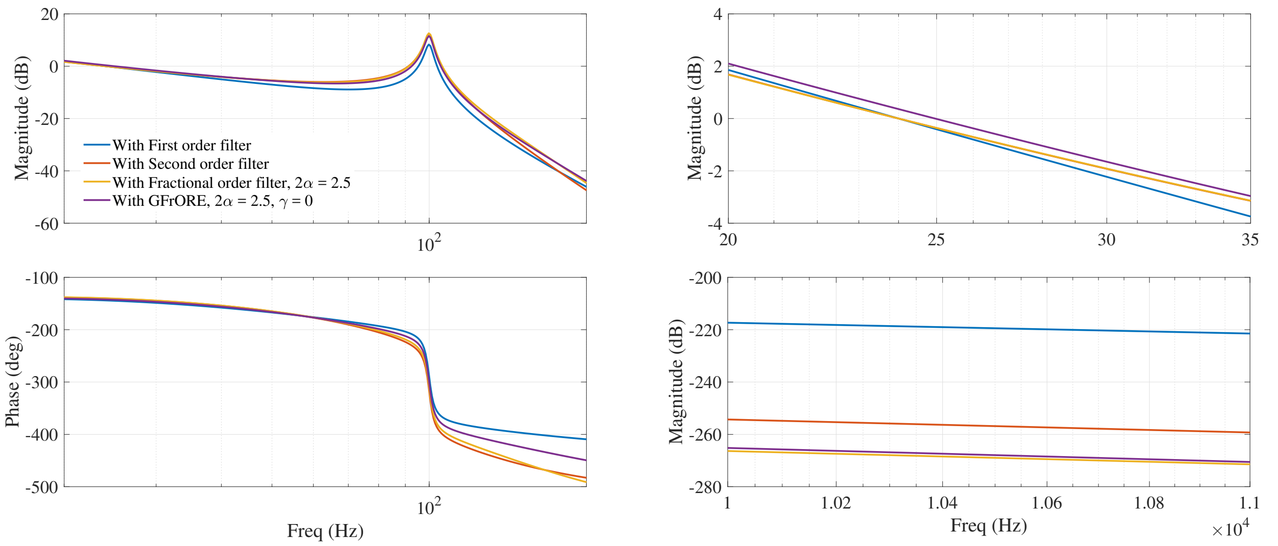

Figure 6: Open loop frequency response for all controller configurations provided. On the left is the overall open loop frequency response. On the right top is zoom in at the bandwidth and on the right bottom is at high frequencies where noise attenuation is necessary

Conclusions

A novel Generalized Fractional Order Reset Element (GFrORE) filter is presented and analysed in this work. The order α provides flexibility over the magnitude slope, while reset action results in reduced phase lag with γ providing the necessary parameter for tuning the same. Reset action however results in the corner frequency being shifted at smaller values of γ and the rule of thumb presented can be used to appropriately design the filter. Further it is seen that reset significantly reduces the Q-factor and indirectly provides greater freedom in shaping phase of filter. A numerical example is provided to highlight the advantage of GFrORE over its linear counterparts.

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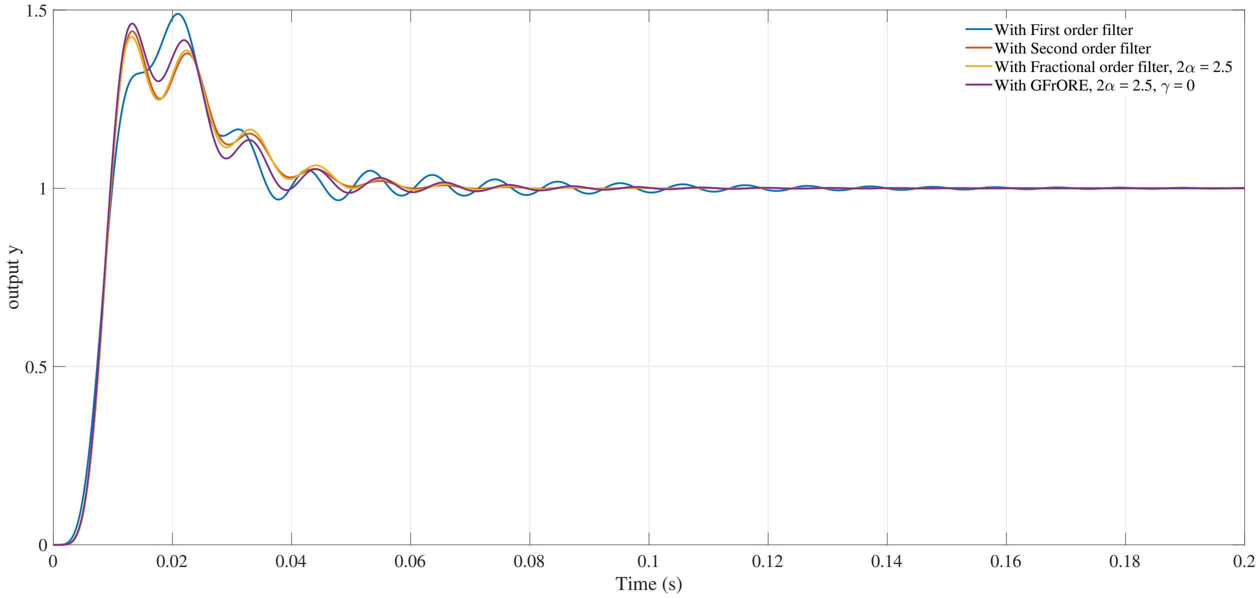

Figure 7: Step response for all controller configurations

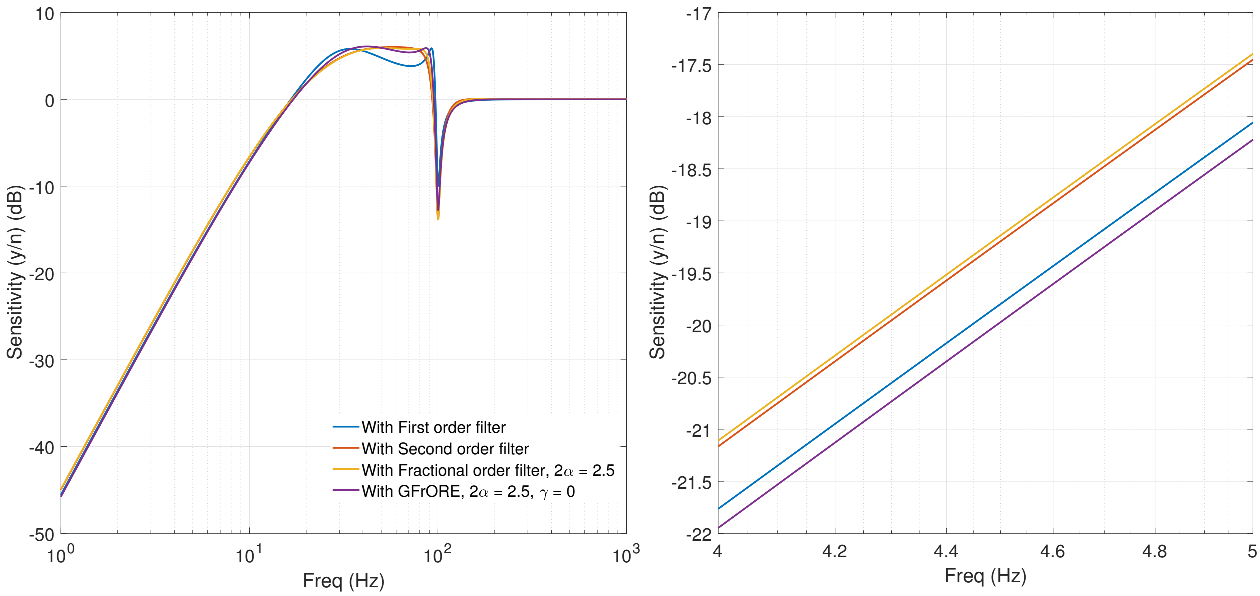

Figure 8: Sensitivity plots for all controller configurations with zoom-in provided at low frequencies

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