Thermodynamical formalism of fractals via Fisher information: Rényi dimensions

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Summary. In a recent paper, an alternative formulation of thermodynamics based on Fisher information has been presented. This is equivalent to the traditional thermodynamics with appropriate Legendre structure. In this presentation this alternative formulation is generalized to chaotic systems. A link between the Rényi dimension and the Fisher informations is explored: the Fisher information can be expressed as a linear combination of the first and second derivatives of the Rényi dimensions with respect to the Rényi parameter and this parameter is taken as the parameter of the Fisher information. A thermodynamical description based on this Fisher information is introduced for chaotic systems. The relationship of the Fisher information and the heat capacity is also emphasized.

I. INTRODUCTION

The Fisher information [1] has a growing emphasis in physics. Frieden [2] described a Legendre transform structure reflecting classical thermodynamics, and the Fisher temperature and Fisher thermodynamics were introduced[3]. An alternative formulation of the thermodynamics based on Fisher information has been presented by Porporato[4]. With a corresponding Legendre structure it gives an identical thermodynamics to the traditional thermodynamics. In this presentation, this alternative formulation based on Fisher information is extended to chaotic systems [5].

II. Formalism

The thermodynamic formalism of chaotic systems[6,7] has proved to be very effective for examining fractal and multifractal objects. The formalism utilizes the analogy of the statistical mechanics and the chaos theory. It has recently been shown that this analogy can be extended by introducing a Fisher information based thermodynamic description. This study provides a new insight into the chaos theory.

The main ingredient of the thermodynamic formalism[6,7] is the $f(\alpha)$ spectrum and its Legendre transform $\tau(\beta)$:

$$\tau(\beta)=\beta\alpha-f(\alpha)=(\beta-1)D(\beta)$$

which is also related to the Rényi dimension $D(\beta)$. The Fisher information is defined as the negative of the second derivative of τ with respect to β:

$$F_\alpha=-\frac{d^2\tau(\beta)}{d\beta^2}=-\frac{\partial\alpha}{\partial\beta^2} \quad .$$

It also gives the variance of α:

$$F_\alpha=\left(\alpha-\langle\alpha\rangle\right)^2=\frac{C(\beta)}{\beta^2}=C(T)T^2 \quad ,$$

where, C is the "heat capacity" of the chaotic system. We have derived a relation between the Fisher information and the Rényi dimension $D(\beta)$:

$$F_\alpha=(1-\beta)\frac{d^2D}{d\beta^2}-2\frac{dD}{d\beta} \quad .$$

We can define the Legendre transform of the Fisher information

$$F_f=\alpha-\beta\frac{d\alpha}{d\beta}=\alpha+\beta F_\alpha$$

with the fundamental relation $\partial F_f/\partial F_\alpha=\beta$.

As an example, the logistic map is presented for the control parameter larger than $r_\infty = 3.56995$, where chaotic behaviour can be observed. Fig. 1. shows the Fisher information F_α and its Legendre transform F_f at β=1.78. The chaotic nature is reflected by a complicated structure in the chaotic regimes. In a periodic window for a period-n orbit, the Renyi dimension is $D = \ln n$, $\tau(\beta) = (\beta - 1)\ln n$ and the Fisher information disappears.

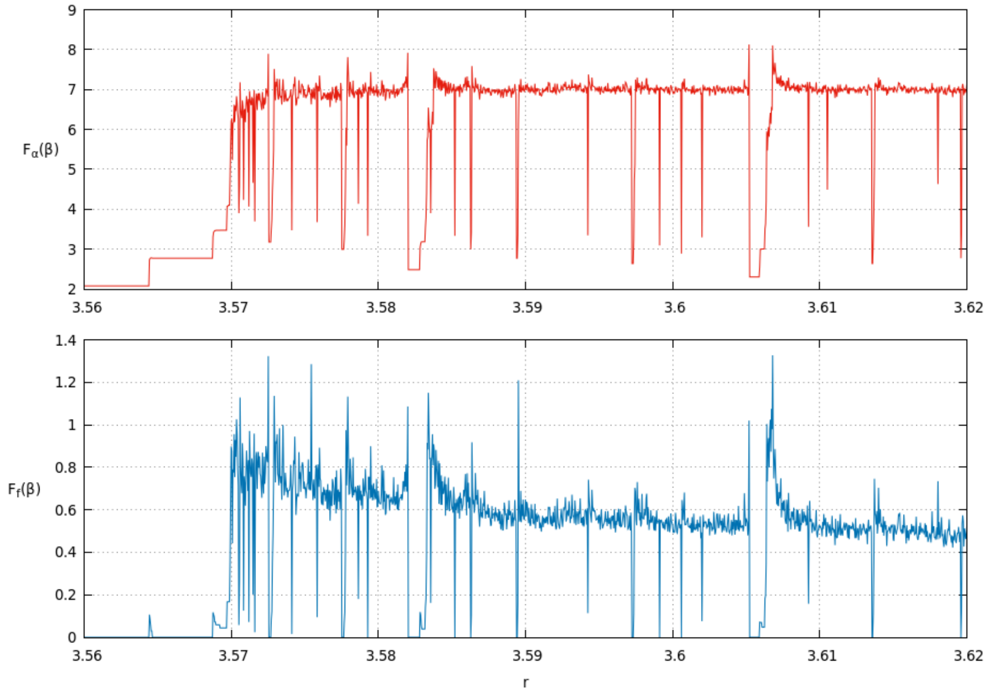

FIG.1. Logistic map: varying the control parameter 'r' shows the structure of the Fisher information $F_\alpha(\beta)$ (top) and its Legendre transforms $F_f(\beta)$ (bottom) at β = 1.78.

III. Conclusions

The Fisher thermodynamical formalism offers an alternative way of studying chaotic systems. Several novel quantities, such as Legendre transforms of Fisher information or heat capacity, provide a new insight in investigating non-linear systems. These quantities are very sensitive to correlations and show deeper description of fluctuations than the usual formalism.

References

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