Towards an Optimal Control Framework for Non-Smooth Mechanical Systems

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Summary. Among a broad range of scientific and engineering problems, non-smooth mechanical systems enduring impact and friction are of great interest. Inherited from non-smoothness, non-differentiability in the equations of motion prevents exploiting the existing methods to design optimal control for such systems. However, adopting tools from convex and non-smooth analysis allows the use of mathematical non-smooth concepts in the traditional optimal control approaches. The purpose of this paper is to delve into the challenges of enhancing the optimal control framework for non-smooth mechanical systems subjected to friction. Thereby, we formulate an optimal control problem for a simple non-smooth mechanical system with Coulomb friction which leads to an impulsive optimal control for the non-smooth system at hand.

We consider a one-dimensional mass-spring-damper system subjected to set-valued Coulomb friction. The dynamics of the system can be given in state-space form as

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) + \mathbf{W}\lambda, \tag{1}$$

where the tuple $\mathbf{x}(t) \in \mathbb{R}^n$ carries the states (displacement and velocity) of the system and $u(t) \in \mathbb{R}$ is the control. Moreover, $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^n$ and $\mathbf{W} \in \mathbb{R}^n$ denote system, input and generalized force direction matrices, respectively. The normalized friction force λ satisfies the inclusion

$$-\lambda \in \mu g \,\mathrm{Sign}(\mathbf{W}^{\mathrm{T}}\mathbf{x}(t)), \tag{2}$$

where $\mathrm{Sign}(\cdot)$ represent the set-valued sign function. In the following, we are interested to design an optimal control with minimum effort. Namely, we are interested in solving the following fixed final time optimal control problem

$$\underset{u}{\text{minimize}} \quad \frac{1}{2}\int_{t_0}^{t_f} u(t)^2 \mathrm{d}t \tag{3a}$$

$$\text{subject to} \quad \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) + \mathbf{W}\lambda \tag{3b}$$

$$-\lambda \in \mu g \,\mathrm{Sign}(\mathbf{W}^{\mathrm{T}}\mathbf{x}(t)) \tag{3c}$$

$$\mathbf{x}(t_0) = \mathbf{x}_0 \quad , \quad \mathbf{x}(t_f) = \mathbf{x}_f. \tag{3d}$$

Due to the friction force inclusion constraint (3c), applying classical optimal control approaches does not lead to an appropriate solution for the existing problem. Accordingly, we need to somehow revise the problem formulation and exploit extended optimal control notions in order to handle the standing constraints. To this end, we first focus on the inclusion (3c). By means of non-smooth and convex analysis (see, e.g., [1, 2]), the set-valued Coulomb friction law can be interpreted as a normal cone inclusion, i.e.,

$$\mathbf{W}^{\mathrm{T}}\mathbf{x}(t) \in \mathcal{N}_C(-\lambda), \tag{4}$$

where $\mathcal{N}_C$ denotes the normal cone to the set $C = [-\mu g, \mu g]$. In a further step, the normal cone inclusion (4) can be written as the equality

$$-\lambda = \mathrm{prox}_C(-\lambda + \mathbf{W}^{\mathrm{T}}\mathbf{x}(t)), \tag{5}$$

in which we employ the proximal point function to replace the inclusion with an equality [1]. Consequently, we deal with the equivalent problem to (3) in which the inclusion constraint (3c) is replaced by (5). It should be noted that the above substitution is rather transforming the inclusion into an equality constraint and does not necessarily overcome the issue of non-differentiability. This, however, affects the dynamics of the costate variable in the sense of Pontryagin's Maximum Principle [3]. It is evident from results of the classical Maximum Principle applied to smooth dynamical systems that the costate dynamic satisfies

$$-\dot{\mathbf{p}}(t)^{\mathrm{T}} = \frac{\partial}{\partial \mathbf{x}}\mathcal{H}(\mathbf{x}(t), \mathbf{p}(t), u(t)), \tag{6}$$

where $\mathbf{p}(t) \in \mathbb{R}^n$ denotes the costate (or adjoint) function and $\mathcal{H}(\mathbf{x}(t), \mathbf{p}(t), u(t))$ is the Hamiltonian function. In this paper, however, we address a non-differentiable Hamiltonian function due to the non-smooth nature of friction. Along these lines, a precise look at the evolution of the costate dynamics while following the Maximum Principle, leads to an impulsive optimal control for the considered non-smooth mechanical system.

References

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[3] Pontryagin, L. S. (1962) The Mathematical Theory of Optimal Processes. John Wiley & Sons, Inc.