Inverse Scattering Problems for the Perturbed Biharmonic Operator

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Summary. Some inverse scattering problems for operator of order 4 which is the perturbation (in smaller terms) of the biharmonic operator are considered. The coefficients of this perturbation are assumed to be from some Sobolev spaces (they might be singular). The classical (as for the Schrödinger operator) scattering theory is developed for this operator of order 4. The inverse scattering problems are considered and their uniqueness is proved. The method of Born approximation and an analogue of Saito's formula are justified for this operator of order 4.

One-dimensional case

The motivation to study nonlinear operators of order four can be found for instance in the theory of vibrations of beams (in the simplest one-dimensional model) and the study of elasticity. Indeed, by looking for the time-harmonic solutions $U(x,t) = u(x)e^{-i\omega t}$ to the nonlinear beam equation

$$\partial_t^2 U(x,t) + \partial_x^4 U(x,t) + mU(x,t) + |U(x,t)|^p U(x,t) = 0, \quad m > 0,$$

we arrive to the equation

$$u^{(4)}(x) + (m + |u|^p)u(x) = \omega^2 u(x), \quad x \in R.$$

We consider one-dimensional quasi-linear 4th order equation of the form

$$L_4 u(x) := u^{(4)}(x) + q_1(x,|u|)u'(x) + q_0(x,|u|)u(x) = k^4 u(x), \quad x \in R,$$

where $u(x)$ denotes, for example, the deflection (displacement) at the point x of the ideal beam, $k \neq 0$ is real number and the potentials $q_1(x,|u|)$ and $q_0(x,|u|)$ are complex-valued (in general) and integrable. By $L^p(R), 1 \le p < \infty$, and $W_1^1(R)$ we denote Lebesgue and Sobolev spaces on the line with the norms

$$\|f\|_{L^p(R)} = \left(\int_{-\infty}^{\infty} |f(x)|^p \, dx \right)^{\frac{1}{p}}, \quad \|f\|_{W_1^1(R)} = \|f\|_{L^1(R)} + \|f'\|_{L^1(R)}.$$

We use also the spaces: $L^\infty(R)$ and $H^t(R)$, with the norms

$$\|f\|_{L^\infty(R)} = ess \sup_{x \in R} |f(x)|, \quad \|f\|^2_{H^t(R)} = \int_{-\infty}^{\infty} (1 + |\xi|^2)^t |Ff(\xi)|^2 \, d\xi,$$

where Ff is the Fourier transform of f, that is

$$Ff(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} f(x) \, dx.$$

In the problems that we consider the main role is played by the special solutions of the equation $L_4 u(x) = k^4 u(x)$, i.e., the solutions of the form (there are also exponentially growing solutions of this equation)

$$u(x,k) = u_0(x,k) + u_{sc}(x,k), \quad u_0(x,k) = e^{ikx},$$

where the scattered part $u_{sc}(x,k)$ satisfy the Sommerfeld radiation conditions at the infinity in the one-dimensional case, i.e.

$$\left(\frac{\partial}{\partial |x|} - ik \right) u_{sc}(x,k) = o(1), \quad |x| \to \infty,$$

$$\left(\frac{\partial}{\partial |x|} - ik \right) (u_{sc}(x,k))''_x = o(1), \quad |x| \to \infty.$$

In that case u_{sc} is the unique solution of the so-called Lippmann-Schwinger integral equation

$$u(x,k) = u_0(x,k) - \int_{-\infty}^{\infty} G_k^+(|x-y|)(q_1(y,|u|)u'(y) + q_0(y,|u|)u(y)) \, dy,$$

where G_k^+ is the outgoing fundamental solution of the one-dimensional Helmholtz operator $\frac{d^4}{dx^4} - k^4$. i.e., the kernel of the integral operator

$$\left(\frac{d^4}{dx^4} - k^4 - i0\right)^{-1} = \frac{1}{2k^2}\left(\left(-\frac{d^2}{dx^2} - k^2 - i0\right)^{-1} - \left(-\frac{d^2}{dx^2} + k^2\right)^{-1}\right).$$

This function $G_k^+(|x|)$ is equal to

$$G_k^+(|x|) = \frac{ie^{i|k||x|} - e^{-|k||x|}}{4|k|^3}.$$

The first and second derivatives of G_k^+ with respect to x can be calculated as

$$(G_k^+(|x|))'_x = \frac{-e^{i|k||x|} + e^{-|k||x|}}{4k^2} sign(x), \quad x \neq 0,$$

$$(G_k^+(|x|))''_x = -\frac{ie^{i|k||x|} + e^{-|k||x|}}{4|k|}, \quad x \neq 0,$$

It can be easily checked also that G_k^+ satisfies for any $k > 0$ the one-dimensional Sommerfeld radiation conditions at the infinity in the form

$$\left(\frac{\partial}{\partial|x|} - ik\right) G_k^+(|x|) = o(1), \quad |x| \to \infty,$$

$$\left(\frac{\partial}{\partial|x|} - ik\right) \left(G_k^+(|x|)\right)''_x = o(1), \quad |x| \to \infty.$$

This function $G_k^+(|x|)$ and its derivatives satisfy also the following uniform estimates

$$|G_k^+(|x|)| \leq \frac{1}{2|k|^3}, \quad |(G_k^+(|x|))'_x| \leq \frac{1}{2k^2}, \quad |(G_k^+(|x|))''_x| \leq \frac{1}{2|k|}.$$

Using these estimates for G_k^+ we prove that for $|k|$ large there is a unique solution of the Lippmann-Schwinger equation and this solution satisfies the estimates

$$\|u - u_0\|_{L^\infty(R)} \leq \frac{c_0}{|k|^2}, \quad \|u' - iku_0\|_{L^\infty(R)} \leq \frac{c_0}{|k|}, \quad u_0 = e^{ikx},$$

uniformly in $|k| \geq c_0$ with $c_0 > 0$ depending on the norms of q_1 and q_0, and admits the following asymptotical representations:

$$u(x,k) = a(k)e^{ikx} + o(1), \quad u(x,k) = e^{ikx} + b(k)e^{-ikx} + o(1), \quad x \to \pm\infty,$$

respectively, where the coefficients $a(k)$ and $b(k)$ are defined as

$$a(k) = 1 - \frac{i}{4k^3}\int_{-\infty}^{\infty} e^{-iky}\left(q_1(y,|u|)u'(y) + q_0(y,|u|)u(y)\right) dy,$$

$$b(k) = -\frac{i}{4k^3}\int_{-\infty}^{\infty} e^{iky}\left(q_1(y,|u|)u'(y) + q_0(y,|u|)u(y)\right) dy$$

and they are called the "transmission" and the "reflection" coefficients, respectively. Defining the solution $u(x,k)$ for negative k as $u(x,k) := \overline{u(x,-k)}$ we obtain that $a(k) = \overline{a(-k)}$ and $b(k) = \overline{b(-k)}$ for negative k. And we put $b(k) = 0$ for $|k| < 2c_0$. Hence, we have well-defined the reflection coefficient $b(k)$ for all $k \in R$. The inverse problem that considered here is to extract some information about the potentials q_0 and q_1 from the knowledge of the reflection coefficient $b(k)$ for $|k|$ arbitrary large.
The properties of $u(x,k)$ allow us to conclude that for $k \to +\infty$

$$b(k) \approx -\frac{i\sqrt{2\pi}}{4k^3}F(\beta)(2k),$$

where $\beta(y) = -\frac{1}{2}q_1'(y,1) + q_0(y,1)$. This asymptotic leads to the direct scattering Born approximation u_B and to the inverse scattering Born approximation q_B, respectively

$$u_B(x,k) = e^{ikx} - \frac{i\sqrt{2\pi}}{4k^3}F(\beta)(2k)e^{-ikx},$$

i.e., we may substitute our scattering solution by this formula which includes only the potential β. The asymptotical representations for $u(x,k)$ for large x can be considered as the analog of the one-dimensional Sommerfeld radiation condition for this operator of order 4. But what is more important, the asymptotic of $u(x,k)$ (or of the reflection coefficient $b(k)$) for large k justifies the following definition which plays the crucial role in the inverse scattering problem.
The inverse scattering Born approximation $q_B(x)$ of the potential β is defined by

$$q_B(x) := F^{-1}\left(\frac{i}{2\sqrt{2\pi}}k^3 b\left(\frac{k}{2}\right)\right),$$

where F^{-1} denotes the inverse Fourier transform on the line and the equality is considered in the sense of tempered distributions.
Denoting $h_1(x) := q_1(x,1)$ and $h_0(x) := q_0(x,1)$ we assume some additional smoothness conditions for q_1 and q_0. Suppose that the following representations hold

$$q_0(x,1+s) = h_0(x) + q_0^*(x,s_0^*)s, \quad q_1(x,1+s) = h_1(x) + q_1^*(x,1)s + q_1^*(x,s_1^{**})\frac{s^2}{2},$$

where $|s_0^*|, |s_1^*| < |s|$, $h_1 \in W_1^1(R)$, $q_1^* \in L^1(R) \cap L^p(R)$ for some $p > 1$, and q_0^* and q_1^* belong to $L^1(R)$ in x uniformly in s, $|s| < s_0$, for some $0 < s_0 < 1$.
The following result is valid: Under the smoothness conditions for q_0 and q_1 mentioned above the inverse scattering Born approximation q_B admits the representation

$$q_B(x) = \Re(\beta)(x) + \frac{1}{\pi} p.v. \int_{-\infty}^{\infty} \frac{\Im(\beta)(y)}{x-y}\, dy \quad (mod \quad C_0(R)),$$

where $C_0(R)$ denotes the space of all continuous functions that vanish at the infinity.
If the potentials q_1 and q_0 are real-valued functions then this result says that

$$q_B(x) - \beta(x) \in C_0(R),$$

that is, all singularities and jumps of the function $-\frac{1}{2}q_1'(x,1) + q_0(x,1)$ can be uniquely determined by the inverse scattering Born approximation with very limited data - we need to know the reflection coefficient (among other four coefficients) only for the value of the spectral parameter k which is arbitrary large.
One more example came from the nonlinear optics. Let us assume that $q_1 \equiv 0$ and q_0 corresponds to the cubic-quintic type of nonlinearity, that is $q_0(x,|u|) = p_1(x)|u|^2 + p_2(x)|u|^4$, where $p_1(x)$ and $p_2(x)$ are unknown functions which are equal to the unknown constants p_1 and p_2 on the unknown interval $[a,b]$ and zero outside of this interval. Then the main result gives us that using inverse scattering Born approximation we may reconstruct this unknown interval $[a,b]$ and the sum $p_1 + p_2$ of these unknown constants p_1 and p_2.

Three-dimensional case

Concerning the scattering theory in three dimensions we have the following. The motivation to study operators of higher order (bigger than 2) appears for example in the study of elasticity and the theory of vibrations of beams. As a concrete example, the beam equation

$$\partial_t^2 U(x,t) + \Delta^2 U(x,t) + mU(x,t) = 0$$

under time-harmonic assumption $U(x,t) = u(x)e^{-i\omega t}$ results in the equation

$$\Delta u(x) + mu(x) = \omega^2 u(x), \quad x \in R^3.$$

Other examples of biharmonic problems include hinged plate configuration, described by equations of the form

$$\Delta^2 u(x) = f(x), \quad x \in \Omega, \quad u(x) = \Delta u(x) = 0, \quad x \in \partial\Omega$$

with the so-called Navier boundary conditions.
We consider the operator of order 4 in the form

$$L_4 u(x) := \Delta^2 u(x) + 2i\vec{W}(x)\nabla u(x) + i(\nabla\vec{W})u(x) + V(x)u(x), \quad x \in R^3,$$

with real-valued functions $\vec{W}$ and V. Our basic assumptions for the coefficients are:

$$\vec{W}(x) \in W^1_{p,\sigma}(R^3), \quad V(x) \in L^p_\sigma(R^3), \quad 3 < p \le \infty, \quad \sigma > \frac{3}{p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

where the weighted Lebesgue space $L^p_\sigma(R^3)$ is defined by the norm

$$\|f\|_{L^p_\sigma(R^3)} := \left(\int\limits_{R^3} (1+|x|)^{\sigma p} |f(x)|^p \, dx \right)^{\frac{1}{p}}$$

and the Sobolev space is defined by $W^1_{p,\sigma}(R^3) := \{f \in L^p_\sigma(R^3) | \nabla f \in L^p_\sigma(R^3)\}$. And by $H^s(R^3)$ we denote L^2-based Sobolev spaces with the smoothness index $s \geq 0$ defined by the norm

$$\|f\|_{H^s(R^3)} := \left(\int\limits_{R^3} (1+|\xi|^2)^s |Ff(\xi)|^2 \, d\xi \right)^{\frac{1}{2}},$$

where F denotes the three-dimensional Fourier transform

$$Ff(\xi) = \int\limits_{R^3} e^{i(x,\xi)} f(x) \, dx,$$

and the weighted Sobolev space $H^s_\delta(R^3)$, $s \geq 0$, we define by $H^s_\delta(R^3) := \{f \in L^2(R^3) | (1+x^2)^\delta f \in H^s(R^3)\}$.
The operator L_4 with these coefficients is symmetric in $L^2(R^3)$, semi-bounded from below and therefore has Friedrichs self-adjoint extension with domain $H^4(R^3)$. We are looking for the scattering solutions of this equation in the form (the same as in one-dimensional case)

$$u(x,k,\theta) = e^{ik(x,\theta)} + u_{sc}(x,k,\theta),$$

where $\theta \in S^2$ is the angle of the incident wave, and $u_{sc}(x,k,\theta)$ satisfies the Sommerfeld radiation conditions at the infinity

$$\left(\frac{\partial}{\partial|x|} - ik \right) u_{sc}(x,k,\theta) = o\left(\frac{1}{|x|}\right), \quad |x| \to \infty,$$

$$\left(\frac{\partial}{\partial|x|} - ik \right) \Delta u_{sc}(x,k,\theta) = o\left(\frac{1}{|x|}\right), \quad |x| \to \infty.$$

These scattering solutions are the unique solutions of the analogue of Lippmann-Schwinger equation for 4th order operator

$$u(x,k,\theta) = e^{ik(x,\theta)} - \int\limits_{R^3} G^+_k(|x-y|) \left(2i\vec{W}(y)\nabla u(y) + \tilde{V}(y)u(y) \right) \, dy,$$

where $\tilde{V} = i\nabla\vec{W} + V$ and the outgoing fundamental solution of three-dimensional operator $\Delta^2 - k^4$ is equal to

$$G^+_k(|x|) = \frac{e^{i|k||x|} - e^{-|k||x|}}{8\pi|x|k^2}.$$

Under the above conditions for $\vec{W}$ and V for $k > 0$ large enough there exists a unique solution $u = u_0 + u_{sc}$, $u_0(x,k,\theta) = e^{ik(x,\theta)}$, of this integral equation such that $u_{sc} = u_{sc}(x,k,\theta)$ belongs to the weighted Sobolev space $H^1_{-\frac{\sigma}{2}}(R^3)$ and it can be obtained as the series of iterations

$$u_{sc}(x,k,\theta) = \sum_{j=0}^{\infty} L_k^{j+1} u_0(x,k,\theta),$$

where L_k denotes the integral operator

$$L_k f(x) = -\int\limits_{R^3} G^+_k(|x-y|) \left(2i\vec{W}(y)\nabla f(y) + \tilde{V}(y)f(y) \right) \, dy.$$

Moreover, the following estimate holds

$$\|u_{sc}\|_{H^1_{-\frac{\sigma}{2}}(R^3)} \leq \frac{C}{k}, \quad k \geq k_0 > 0,$$

with k_0 large enough and with constant $C > 0$ which depends only on the corresponding norms of $\vec{W}$ and V. This function u_{sc} actually belongs to the space $H^2_{-\frac{\sigma}{2}}(R^3)$ with uniformly bounded in $k \geq k_0$ norm in this space.
The conditions for $\vec{W}$ and V allow us to obtain the following asymptotical as $|x| \to \infty$ representation ($k \geq k_0$ and fixed):

$$u(x,k,\theta) = e^{ik(x,\theta)} - \frac{e^{ik|x|}}{8\pi|x|} A(k,\theta,\theta') + o\left(\frac{1}{|x|}\right), \quad |x| \to \infty,$$

where $\theta' = \frac{x}{|x|}$ is the angle of observation. Here the function $A(k,\theta,\theta')$ is called the scattering amplitude and is defined by

$$A(k,\theta,\theta') = \frac{1}{k^2}\int\limits_{R^3} e^{-ik(y,\theta')}\left(2i\vec{W}(y)\nabla u(y,k,\theta) + \tilde{V}(y)u(y,k,\theta)\right) dy.$$

The conditions for the coefficients $\vec{W}$ and V guarantee that this function $A(k,\theta,\theta')$ is continuous and bounded with respect to all its arguments k,θ,θ'. And this function $A(k,\theta,\theta')$ gives us the data for the inverse scattering problems. The first result is the analogue of Saito's formula for 4th order operator.
If $\vec{W} \in W^1_{p,\delta}(R^3)$ and $V \in L^p_\delta(R^3)$ where $3 < p \le \infty$ and $\delta > 3 - \frac{3}{p}$, then the limit

$$\lim_{k\to+\infty} k^4 \int\limits_{S^2\times S^2} e^{-ik(\theta-\theta',x)}A(k,\theta,\theta')\,d\theta\,d\theta' = 8\pi^2\int\limits_{R^3}\frac{V(y)}{|x-y|^2}\,dy$$

holds uniformly in $x \in R^3$.
An important consequence of Saito's formula is the following uniqueness result for the inverse scattering problem with full data.
Let $\vec{W}_1, V_1$ and $\vec{W}_2, V_2$ be as before. If the corresponding scattering amplitudes $A_1(k,\theta,\theta')$ and $A_2(k,\theta,\theta')$ coincide for some sequence $k_j \to +\infty$ and for all angles $\theta,\theta' \in S^2$ then the coefficients V_1 and V_2 are equal a.e. Even more is true - under the same assumptions as in Saito's formula we have the following representation

$$V(x) = \frac{1}{16\pi^4}\lim_{k\to+\infty} k^4 \int\limits_{S^2\times S^2} e^{-ik(\theta-\theta',x)}A(k,\theta,\theta')|\theta-\theta'|\,d\theta\,d\theta'$$

that must be understood in the sense of tempered distributions.
As a different data for the reconstruction of unknown potential $V(x)$ (the function $\vec{W}$ in the operator L_4 might be arbitrary from the space $W^1_{p,\sigma}(R^3)$ in that case) we consider the kernel $G_p(x,y,k)$ of the integral operator $(L_4 - k^4 - i0)^{-1}$ which can be obtained as the solution of the integral equation

$$G_p(x,y,k) = G_k^+(x,y,k) - \int\limits_{R^3} G_k^+(x,z,k)\left(2i\vec{W}(z)\nabla_z G_p(z,y,k) + \tilde{V}(z)G_p(z,y,k)\right) dz.$$

The solvability of this equation can be obtained by the same manner as the solvability of the Lippmann-Schwinger equation for the scattering solutions. As the result, we have that

$$\|(L_4 - k^4 - i0)^{-1}f\|_{H^1_{-\frac{\sigma}{2}}(R^3)} \le \frac{C}{k^2}\|f\|_{L^2_{\frac{\sigma}{2}}(R^3)}, \quad k \ge k_0 > 0,$$

where σ is as before. This fact implies that $G_p(x,y,k)$ has the same estimates as $G_k^+(x,y,k)$ since it can be obtained as the series of iterations of G_k^+. More precisely, the following uniform estimate (with respect to $x,y \in R^3$ and $k \ge k_0 > 0$ with k_0 large enough) holds

$$|G_p(x,y,k) - G_k^+(|x-y|)| \le \frac{C}{k^3}.$$

The knowledge of the function $G_p(x,y,k)$ for large values of k and the fact that this function solves the corresponding integral equation allow us to calculate at every (fixed) point ξ the Fourier transform of V by the formula

$$F(V)(\xi) = \lim_{x,y\to\infty,k\to+\infty} 64\pi^2 k^4|x||y|e^{-ik(|x|+|y|)}\left(G_k^+(|x-y|) - G_p(x,y,k)\right),$$

where $\xi = -k\left(\frac{x}{|x|} + \frac{y}{|y|}\right)$ (fixed) and where F denotes the three-dimensional Fourier transform. Here we assumed that either $\vec{W}$ and V have compact support or have some special behavior at the infinity like $O(|x|^{-\mu})$ with some $\mu > 3$. This result implies one more uniqueness result in the inverse scattering problem. Namely, if $G_p^{(1)}(x,y,k)$ and $G_p^{(2)}(x,y,k)$ are two different kernels which correspond to two different pairs of the coefficients $\vec{W}_1, V_1$ and $\vec{W}_2, V_2$, and if $G_p^{(1)}(x,y,k) = G_p^{(2)}(x,y,k)$ for all $x,y,k \to \infty$, then $V_1(x) = V_2(x)$ a.e.
Our next steps are devoted to the considerations of the direct and inverse scattering Born approximation. Substituting $u = u_0 + u_{sc}$ into the scattering amplitude A gives that

$$A(k,\theta,\theta') = \frac{1}{k^2}\int\limits_{R^3} e^{-ik(y,\theta')}\left(2i\vec{W}(y)\nabla u_0(y,k,\theta) + \tilde{V}(y)u_0(y,k,\theta)\right) dy +$$

$$+\frac{1}{k^2}\int\limits_{R^3} e^{-ik(y,\theta')}\left(2i\vec{W}(y)\nabla u_{sc}(y,k,\theta) + \tilde{V}(y)u_{sc}(y,k,\theta)\right) dy =: A_B(k,\theta,\theta') + R(k,\theta,\theta').$$

The function $A_B(k,\theta,\theta')$ is called the direct Born approximation. It can be easily checked that $A_B(k,\theta,\theta')$ is actually equal to

$$A_B(k,\theta,\theta') = -\frac{2\theta}{k}F(\vec{W})(k(\theta-\theta')) + \frac{1}{k^2}F(i\nabla\vec{W}+V)(k(\theta-\theta')) =$$
$$= -\frac{\theta+\theta'}{k}F(\vec{W})(k(\theta-\theta')) + \frac{1}{k^2}F(V)(k(\theta-\theta')),$$

and the rest term $R(k,\theta,\theta')$ might be estimated appropriately.
The direct Born approximation can solve (somehow) the problem of reconstruction unknown coefficients V and $\vec{W}$ in the following sense.
Let $\xi \neq 0$ be an arbitrary vector from R^3 and let ω be the unit vector that is orthogonal to ξ. Let also $k > 0$ be so that $\xi^2 \leq 4k^2$. If we chose θ and θ' such that

$$\theta = \frac{\xi}{2k} + \frac{\omega}{2k}\sqrt{4k^2-\xi^2}, \quad \theta' = -\frac{\xi}{2k} + \frac{\omega}{2k}\sqrt{4k^2-\xi^2}$$

then $\theta,\theta' \in S^2, \xi = k(\theta-\theta')$ and

$$F(V)(\xi) = \frac{k^2}{2}\left(A_B(k,\theta,\theta') + A_B(k,-\theta',-\theta)\right),$$

$$\sqrt{4k^2-\xi^2}(F(\vec{W})(\xi),\omega)_{R^3} = \frac{k^2}{2}\left(A_B(k,\theta,\theta') - A_B(k,-\theta',-\theta)\right).$$

These equations give us clearly solution for V and $curl\vec{W}$.
Our next main interest (with respect to real inverse problems) concerns to the particular case $\theta' = -\theta$. This case leads to the so-called direct backscattering Born approximation, i.e.

$$A_B^b(k,\theta,-\theta) = \frac{1}{k^2}F(V)(2k\theta), \quad A(k,\theta,-\theta) \approx \frac{1}{k^2}F(V)(2k\theta).$$

This direct approximation justifies the following inverse backscattering Born approximation.
The inverse backscattering Born approximation $V_B^b(x)$ in the operator L_4 is defined as

$$V_B^b(x) = \frac{1}{(2\pi)^3}\int_0^\infty k^4\,dk\int_{S^2} e^{-ik(x,\theta)}A\left(\frac{k}{2},\theta,-\theta\right)d\theta,$$

where the equality is understood in the sense of tempered distributions. Due to this definition and the definition of the scattering amplitude we may conclude that

$$V_B^b(x) = V(x) + V_1(x) + V_{rest}(x),$$

where the quadratic form $V_1(x)$ (first nonlinear term in the inverse backscattering Born approximation) can be calculated precisely and it is equal to

$$V_1(x) = -\frac{4}{(2\pi)^3}F^{-1}_{\xi\to x}\left(\int_{R^3}\frac{F(\overline{\tilde{V}})(\xi-\eta)F(\tilde{V})(\eta)}{\xi^2(\eta^2-(\eta,\xi)-i0)(\eta^2-(\eta,\xi)+\frac{\xi^2}{2})}d\eta\right) +$$
$$+\frac{1}{(2\pi)^3}F^{-1}_{\xi\to x}\left(\int_{R^3}\frac{\xi F(\vec{W})(\xi-\eta)(\xi+\eta)F(\vec{W})(\eta)}{\xi^2(\eta^2-(\eta,\xi)-i0)(\eta^2-(\eta,\xi)+\frac{\xi^2}{2})}d\eta\right).$$

Here F^{-1} denotes the inverse Fourier transform in R^3 and $\overline{\tilde{V}}$ is the complex conjugate of $\tilde{V} = i\nabla\vec{W} + V$. This precise formula for V_1 and the mentioned above conditions for the coefficients V and $\vec{W}$ of the operator L_4 allow us to prove that V_1 (as a function of x) is actually continuous function. Concerning the rest term V_{rest} using the estimates for u_{sc} we obtain that it belongs to the Sobolev space $H^t(R^3)$ with any $t < \frac{3}{2}$. Thus, we have that

$$V_B^b(x) - V(x) \in H^t_{loc}(R^3), \quad t < \frac{3}{2},$$

i.e., this difference belong to the "smoother" space than L^p. This fact means that using the inverse backscattering Born approximation $V_B^b(x)$ we can reconstruct all local singularities from $L^p(R^3)$ of the unknown potential $V(x)$ (we note that $\vec{W}(x)$ is continuos by the assumptions) for any $3 < p < \infty$.

Conclusions

It is shown that the classical inverse scattering Born approximation efficiently works for higher order (and even quasi-linear) operators with singular coefficients. In particular:

- In one-dimensional case we consider the quasi-linear perturbation of the biharmonic operator and obtained that all singularities and jumps of the potential $\beta(x) = -\frac{1}{2}q_1'(x,1) + q_0(x,1)$, where q_0, q_1 are quasi-linear coefficients of the perturbed operator, can be uniquely determined using only the reflection coefficient for arbitrary large spectral parameter $k > 0$.
- In three-dimensional case we consider the first order perturbation of the biharmonic operator and obtained the analogue of the classical Saito's formula together with uniqueness result and with representation formula. These formulas use only the scattering amplitude with large spectral parameter k as in one-dimensional case. In addition we are able to formulate the inverse backscattering Born approximation for this type of fourth order operator and provide the reconstruction of singularities via this Born approximation.
- New data for the inverse scattering problem in three-dimensional case is considered. Namely, it is proved that the knowledge of the kernel $G_p(x,y,k)$ of the integral operator $(\Delta^2 + 2i\vec{W}(x)\nabla + i\nabla\vec{W}(x) + V(x) - k^4 - i0)^{-1}$ for large values x, y, k uniquely determines the unknown potential V. Moreover, we obtained the effective formula for computing this unknown potential V using this new data.