Comparison between piecewise linear and smooth dynamics: A case study of decomposing a degenerate bifurcation

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Summary. A cyclic system of eight autonomous ordinary differential equations with two–sided, asymmetric couplings is considered. We focus on the emergence and on the dynamical structure of the closure of the union of unstable sets belonging to inhomogeneous equilibria, a two–dimensional sphere. Results for the standard saturated piecewise linear activation function σ_p and for the smooth, tangent hyperbolic sigmoid nonlinearity σ_s are compared.

We consider the cyclic system of differential equations

$$\dot{x}_n = -x_n + \alpha\sigma(x_{n-1}) + \beta\sigma(x_{n+1}) \quad , \qquad n = 1, 2, \ldots, N = 8 \tag{1}$$

with parameters in $\mathcal{R} = \{(\alpha, \beta) \in \mathbb{R}^2 \mid \alpha > 1 \text{ and } \beta \in [0, \alpha]\}$ equipped with the piecewise linear as well as with the tangent hyperbolic sigmoid nonlinearity

$$\sigma(x) = \sigma_p(x) = 2^{-1}(|x+1| - |x-1|) \qquad \text{and} \qquad \sigma(x) = \sigma_s(x) = \tanh(x) \quad , \qquad x \in \mathbb{R}\,.$$

It is clear that the phase portrait of (1) is invariant with respect to the reflection operator $\rho : \mathbb{R}^8 \to \mathbb{R}^8$, $x \to \rho(x) = -x$ and the right shift cyclic permutation operator $\pi : \mathbb{R}^8 \to \mathbb{R}^8$, $x = (x_1, x_2, \ldots, x_7, x_8) \to \pi(x) = (x_8, x_1, x_2, \ldots, x_7)$.

We start with the equilibrium point analysis of the piecewise linear case $\sigma = \sigma_p$. There are two bifurcation curves $\mathcal{C}_\ell = (\alpha, \beta_\ell(\alpha)) \in \mathcal{R}$ where $\beta_\ell(\alpha) = \frac{1+\alpha^2}{1+\alpha}$ for $\alpha > 1$ and $\mathcal{C}^u = (\alpha, \beta^u(\alpha)) \in \mathcal{R}$ where β^u is defined for $\alpha > \frac{1+\sqrt{5}}{2}$. It is not hard to show that both β_ℓ and β^u are smooth functions with positive derivatives and they satisfy the chain of inequalities $\alpha - 1 < \beta_\ell(\alpha) < \beta^u(\alpha) < \alpha$ on the respective domains. We have no explicit formula for $\beta^u = \beta^u(\alpha)$ which is a solution branch of equation $\alpha^3\beta - \alpha^2\beta^2 - \alpha^2 + 3\alpha\beta - \beta^3 - 1 = 0$. (It is worth mentioning that both β_ℓ and β^u are the same for each $N = 2M$, $M \geq 4$ [1], [2]. The main fields investigated in the papers above are metastable periodic rotating waves and their heteroclinic bifurcations as $\beta \to \beta_\ell(\alpha)$ from below and not equilibrium point analysis.) For $(\alpha, \beta) \in \mathcal{R}$, system (1) has three homogeneous equilibria $\{\xi^\pm, \mathbf{0}\} \subset \mathbb{R}^8$ with coordinates $\pm(\alpha + \beta)$ and 0, respectively. The nonzero homogeneous equilibria are asymptotically stable. For $\alpha > 1$ and $0 < \beta < \beta_\ell(\alpha)$, system (1) has no further equilibria.

The piecewise linearity of $\sigma = \sigma_p$ makes brute force computations in (1) possible. Equilibrium point analysis is particularly simple and requires only the solution of linear systems of algebraic equations. Also the eigenstructure of equilibria satisfying $x_n \neq \pm 1$, $n = 1, 2, \ldots, 8$ can be easily obtained by linear algebra, abstract and computational. Forgetting about the origin, all eigenvalues are real and the number of unstable eigenvalues is at most two. This leads to a complete description of equilibrium bifurcations and also to a complete description of orbit connections between equilibria. All the three Figures below portray the dynamical structure of the closure of the union of unstable sets belonging to inhomogeneous equilibria, a two–dimensional topological sphere $\mathbb{S}^2$ with equilibrium patterns near the "equator" and with two additional homogeneous equilibria ξ^+ and ξ^- as "north pole" and "south pole", respectively.

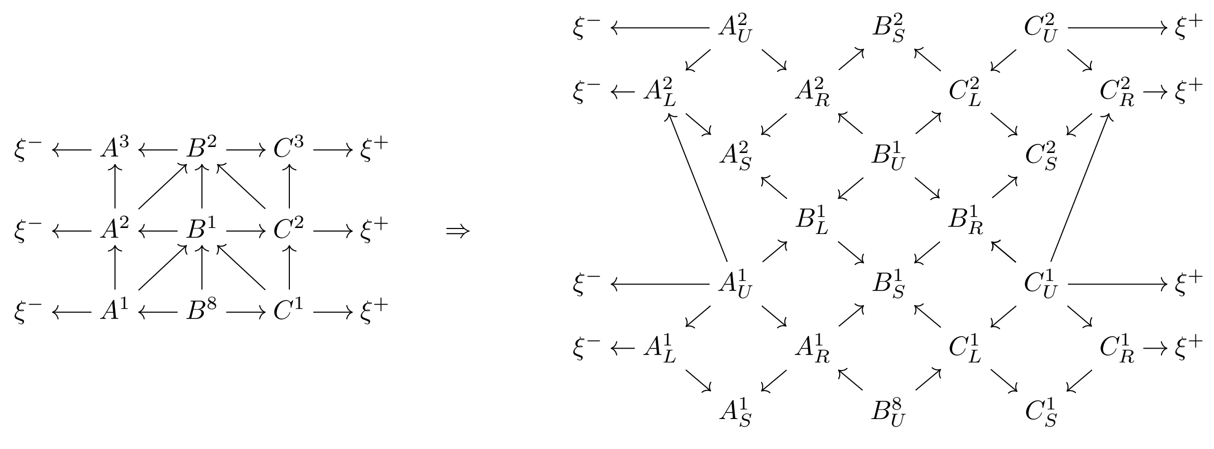

Figure 1: Part of the dynamics at the moment of the first bifurcation at $\beta = \beta_\ell(\alpha)$ and thereafter, for parameters $\beta_\ell(\alpha) < \beta < \beta^u(\alpha)$ where $\alpha > \frac{1+\sqrt{5}}{2}$ is fixed. The total numbers of nonhomogeneous equilibria are $8 \times 3 = 24$ and $24 \times 4 = 96$, respectively.

For brevity, we set $\gamma = \alpha + \beta$. The non–homogeneous equilibria belonging to parameters on $\mathcal{C}_\ell$ are given as

$$\begin{aligned} A^1 &= \big(-\gamma, -\gamma, -\alpha\beta - \alpha + \beta^2, -\alpha + \beta, 1, \alpha\beta + \alpha - \beta^2, \alpha - \beta, -1\big) \\ B^1 &= \big(-1, -\gamma, -\alpha\beta - \alpha + \beta^2, -\alpha + \beta, 1, \gamma, \alpha\beta + \alpha - \beta^2, \alpha - \beta\big) \\ C^1 &= \big(-1, -\alpha\beta - \alpha + \beta^2, -\alpha + \beta, 1, \gamma, \gamma, \alpha\beta + \alpha - \beta^2, \alpha - \beta\big) \end{aligned}$$

and their $X^{k+1} = \pi(X^k)$, $X = A, B, C$, $k = 1, 2, \ldots, 8$ counterparts via cyclic permutations, respectively. Nonhomogeneous equilibria portrayed undergo the same, highly degenerate bifurcation. For example, C^2 gives rise to the source C^2_U, the sink C^2_S, and the two saddles C^2_R, C^2_L. Arrows between equilibria indicate orbit connections. (Concepts of two–dimensional dynamics are understood within the two–dimensional topological sphere, invariant with respect to the global dynamics as well as under the symmetry operators π and ρ. Except the origin—and up to the right shift cyclic permutation operator π—all equilibria and all orbit connections between equilibria are presented. The reflection operator ρ does not lead to new equilibria.) The new equilibria belonging to parameters on $\mathcal{C}^u$ are given as

$$\begin{aligned} D^1 &= \big(-\gamma, -\gamma, -\alpha\beta - \alpha + \beta^2, -\alpha + \beta, 1, \tfrac{-\alpha+\beta^2}{\alpha\beta-1}, \tfrac{-\alpha^2+\beta}{\alpha\beta-1}, \tfrac{-\alpha^3+\alpha\beta}{\alpha\beta-1} - \beta\big) \in \textsc{Separatrix}(A^1_U \to A^2_L) \\ E^1 &= \big(-1, \tfrac{\alpha-\beta^2}{\alpha\beta-1}, \tfrac{\alpha^2-\beta}{\alpha\beta-1}, \tfrac{\alpha^3-\alpha\beta}{\alpha\beta-1} + \beta, \gamma, \gamma, \gamma, \alpha\beta + \alpha - \beta^2, \alpha - \beta\big) \in \textsc{Separatrix}(C^1_U \to C^2_R) \end{aligned}$$

and their $X^{k+1} = \pi(X^k)$, $X = D, E$ and $k = 1, 2, \ldots, 8$ counterparts via cyclic permutations, respectively. Thus the number of new equilibria belonging to parameters on $\mathcal{C}^u$ is 16 (here again, reflection ρ does not lead to new equilibria).

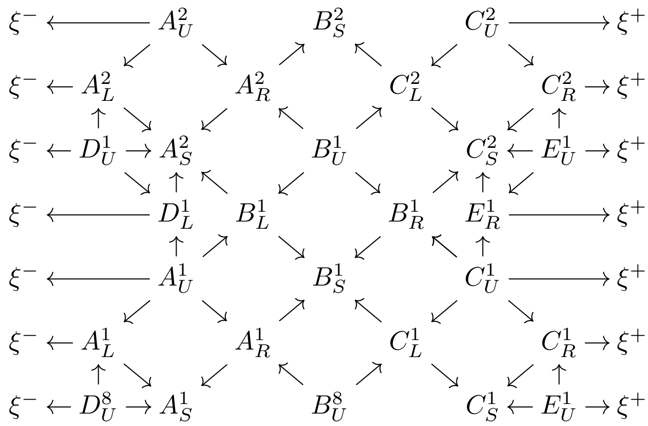

Figure 2: Part of the dynamics after the second, fold bifurcation at $\beta = \beta^u(\alpha)$ where $\alpha > \frac{1+\sqrt{5}}{2}$ is fixed. Observe that equilibrium E^1 splits into the saddle E^1_R and the source E^1_U. Thus the total number of nonhomogeneous equilibria is $96 + 16 \times 2 = 128$.

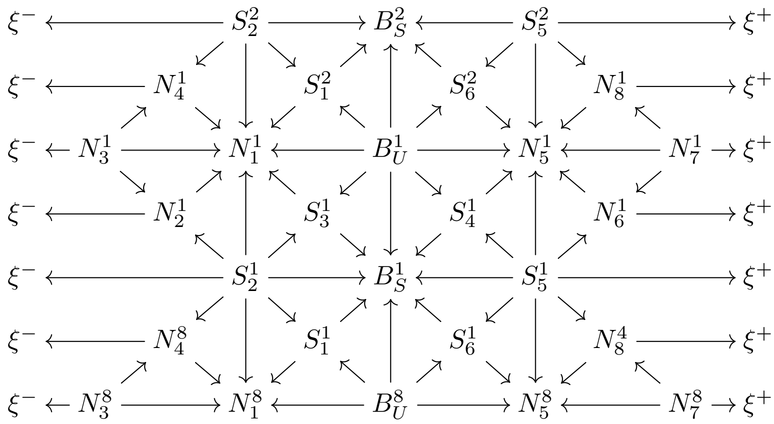

Figure 3: The final conclusion of the *MatCont* analysis of system (1) in the smooth case $\sigma = \sigma_s$: after the last bifurcation, the two flows restricted to the respective two–dimensional invariant spheres are conjugate. While keeping the symmetry properties, the degenerate first bifurcation is decomposed into the four–member bifurcation series *heteroclinic with simultaneous fold, pitchfork, fold, fold.*

References

[1] Forti M., Garay B.M., Koller M., Pancioni,L. (2015) Long transient oscillations in a class of cooperative cellular neural networks. *Int. J. Circuit Theory Applications* **43**:635–655.

[2] DiMarco M., Forti M., Garay B.M., Koller M., Pancioni,L. (2016) Floquet multipliers of a metastable rotating wave in a Chua–Yang ring network, *J. Math. Anal. Appl.* **434**:798–836.