On periodic trajectories of a near-Hamiltonian autonomous dynamical system

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Summary. A non-conservative near-Hamiltonian autonomous dynamical system of the second order is studied. Conditions of existence of periodic trajectories are derived basing on the Poincare-Pontryagin generating function I. For neighbourhoods of fixed points of the generating Hamiltonian system, the "second iteration" I_2 of the Poincare-Pontryagin function is constructed. Simple roots of any of functions I and I_2 correspond to rough trajectories of the initial conservative system. Examples of systems are provided, for which sets of periodic trajectories corresponding to simple roots of these two functions complement each other.

Statement of the problem

An autonomous dynamical system with one degree of freedom with cylindrical phase space is studied. Kinetic and potential energy of the system are $T = 0.5a_1 k(\varphi)\dot{\varphi}^2$ and $U = a_2 u(\varphi)$ respectively, where φ is the angular coordinate; $k(\varphi) > 0$, $u(\varphi) \geq 0$ – dimensionless functions; a_1, a_2 – positive dimensional coefficients.

Assume that the system is near-Hamiltonian. Then equations of motion of the system can be represented in the following dimensionless form:

$$\frac{d\varphi}{d\tau} = \frac{\partial H_0}{\partial p}; \qquad \frac{dp}{d\tau} = -\frac{\partial H_0}{\partial \varphi} + \varepsilon Q(\varphi, p). \qquad H_0 = 0.5k^{-1}(\varphi)p^2 + u(\varphi). \tag{1}$$

Here $Q(\varphi, p)$ corresponds to the generalized non-conservative force, ε is a small positive parameter. Denote $H_0(\varphi, p) = h$ the energy level of the generating Hamiltonian system H_0. Assume that the right-hand part of (1) is analytical with respect to φ and p, and 2π–periodic with respect to φ.

The problem is to describe periodic trajectories (2π–periodic or cycles) that exist in the system (1) for sufficiently small ε, at least, to estimate the number of such trajectories. The Poincare-Pontryagin approach provides sufficient conditions of existence of periodic trajectories in the system (1).

Comparison of sufficient and necessary conditions of existence of periodic trajectories

Let us introduce the following notations: $f(h,\varphi) = \pm\sqrt{2(h-u(\varphi))k(\varphi)}$ for $h \geq \min(u(\varphi))$, $\varphi \in \{\varphi : h \geq u(\varphi)\}$; E is the set of values of h, for which the curve $f(h,\varphi)$ contains any points with $\partial H_0 / \partial\varphi = \partial H_0 / \partial p = 0$.

The Poincare-Pontryagin function $I(h)$ [1] is the average value of the function $Q(\varphi, p)$ along the curve $f(h_0,\varphi)$.

The Poincare-Pontryagin theorem states that if $h_0 \notin E$ is a simple root of $I(h)$, then for sufficiently small ε the system (1) possesses a *rough* periodic trajectory emerging from the trajectory $p = f(h_0,\varphi)$ of H_0 [1, 2].

If the right-hand part of (1) is polynomial, the traditional approach to estimating the number of periodic trajectories of (1) is to find the number of simple roots of $I(h)$ [3].

However, for a non-polynomial system, one can show that special effects take place for $h_0 \in E$:

If $h_0 \in E$ is a root of $I(h)$, then the system (1) may have *one or several rough* periodic phase trajectories in a small neighborhood of $p = f(h_0,\varphi)$. However, (1) may have no periodic trajectory in this domain.

Moreover, the following can be proved: the necessary condition of existence of a periodic trajectory of the system (1) in a small neighborhood of the curve $f(h_0,\varphi)$ for small ε is that $h_0 \in E \cup \{h : I(h) = 0\}$.

Example of "additional" periodic trajectories corresponding to the set *E*

Equations of motion of an aerodynamic pendulum with vertical axis of rotation in a steady wind flow (fig. 1*a*) can be written in the form (1) with $H_0 = 0.5p^2$, $Q(\varphi, p) = \sqrt{(p+\sin\varphi)^2 + \cos^2\varphi}(C_y(\alpha)\cos\varphi - C_x(\alpha)(p+\sin\varphi)) - bp$, $\varepsilon = 0.5\rho S r^3 J^{-1}$, where $\alpha = \operatorname{arctg}(\cos\varphi / (p+\sin\varphi))$ is the instantaneous angle of attack, $C_x(\alpha) = 0.1 + \sin^2\alpha$, $C_y(\alpha) = \sin 2\alpha$ are drag and lift aerodynamic coefficients, b is a viscous friction coefficient in the shaft, ρ is the air density, r is the length of the holder OA, S is the wing area. The task is to find autorotations with positive values of p and auto-oscillations of the system that exist for small ε.

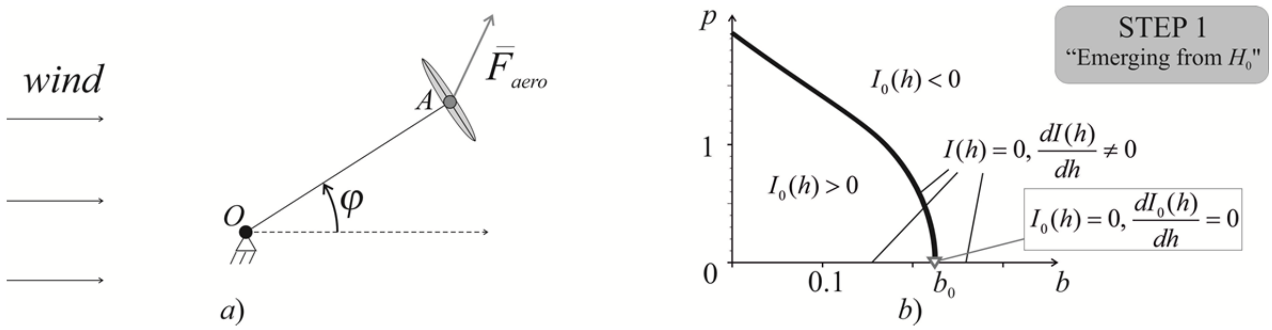

Fig. 1 *a*) An aerodynamic pendulum (top view); *b*) Bifurcation diagram of periodic trajectories emerging from trajectories of H_0.

The function $I(h)$ takes the form: $I(h) = \frac{1}{2\pi}\int_0^{2\pi} Q(\varphi, \sqrt{2h})d\varphi$. For each value of *b*, simple roots of $I(h)$ can be found numerically. The value $p|_{\varphi=\pi}$ for the corresponding periodic trajectory of (1) for $\varepsilon \to 0$ can be characterized by the value $f(h,0) = \sqrt{2h}$. A solid curve in Fig. 1*b* shows the corresponding bifurcation diagram of 2π–periodic trajectories of (1) emerging from trajectories of the system H_0 at $\varepsilon \to 0$ for different *b* (simple roots of $I(h)$ correspond to the solid curve and to the line $p = 0$, except the point $\{b_0, 0\}$ that corresponds to a multiple root). The set *E* for this system is $\{h = 0\}$ and corresponds to the line $p = 0$.

To examine existence of "additional" periodic trajectories of (1) corresponding to the set *E*, perform a change of variables and time: $y = p/\sqrt{\varepsilon}$, $s = \tau\sqrt{\varepsilon}$. The system (1) for small *p* takes the form:

$$\frac{d\varphi}{ds} = \frac{\partial H_1}{\partial y}; \qquad \frac{dy}{ds} = -\frac{\partial H_1}{\partial \varphi} + \sqrt{\varepsilon}\left(y \left.\frac{\partial F(\varphi, p)}{\partial p}\right|_{p=0} - by \right) + O(\varepsilon), \qquad H_1 = \frac{y^2}{2} + \int_{\varphi}^{\pi} Q(\vartheta, 0)d\vartheta. \tag{2}$$

The system H_1 locally „approximates" the system (1) for sufficiently small *p*. The function $u(\varphi)$ for the system H_1 and the qualitative phase portrait of the system H_1 are shown in Fig. 2*a*. Construct a new generating function $I_2(h)$, roots of which correspond to rough periodic trajectories of (2). For each value of *b*, simple roots of $I_2(h)$ can be found numerically. A solid curve at Fig. 2*b* shows the bifurcation diagram of 2π–periodic trajectories and cycles (which cross the line $\varphi = \pi$) of (1) that emerge from trajectories $f_2(h, \varphi)$ of the system H_1 at $\varepsilon \to 0$.

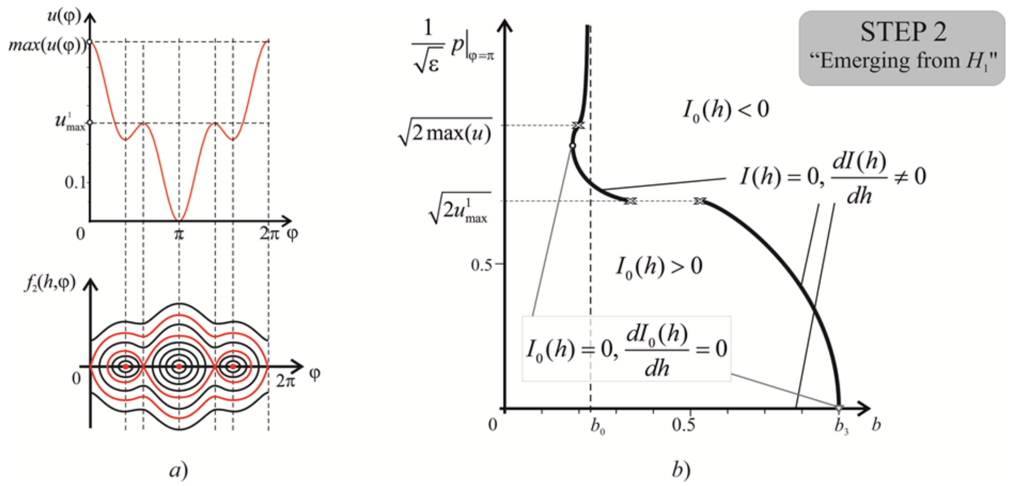

Fig. 2 *a*) On a phase portrait of the system H_1; *b*) Bifurcation diagram of periodic trajectories emerging from trajectories of H_1.

Comparing fig. 1*b* and fig. 2*b*, one can see, in particular, that a rough cycle is detected for $b > b_0$ at the second "iteration" of the Poincare-Pontryagin approach (not at the first).

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References

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