Modeling of the Dynamics of an Autoparametric System with the Spherical Pendulum

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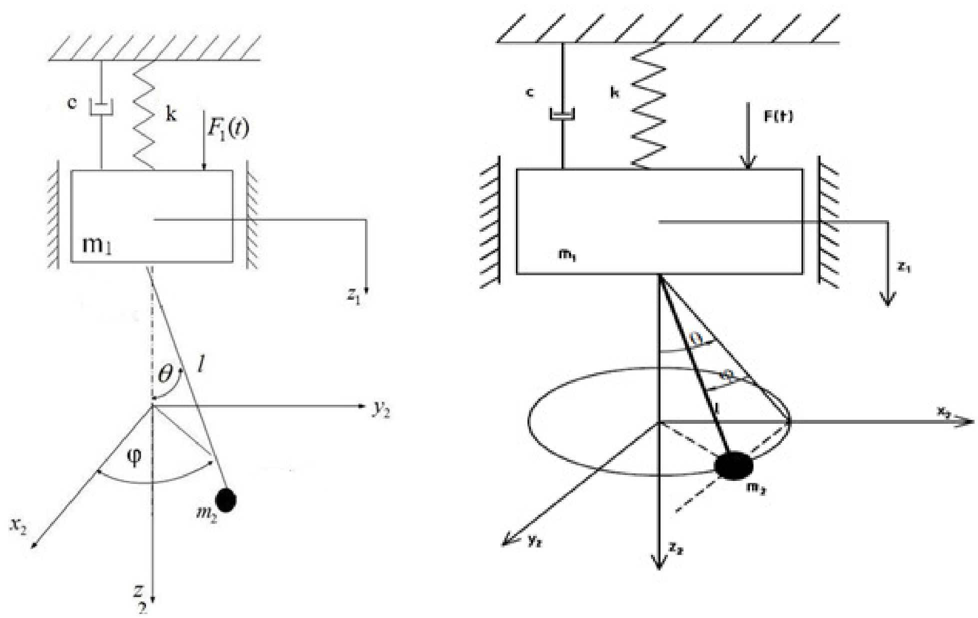
Summary. Dynamic properties of the three degrees of freedom autoparametric system with spherical pendulum are investigated. It was assumed that the spherical pendulum is suspended to the oscillator excited harmonically in the vertical direction. Two models of spherical pendulum are studied: first by angles typical for spherical pendulum and second by angles in the vertical plane. Dynamic properties of the system described by three differential equations containing strongly nonlinear terms are investigated numerically. In autoparametric system one mode of vibration may excite or damp another one, and for except periodic or quasi-periodic vibrations there may also appear chaotic vibration.

Introduction

Oscillations of an autoparametric system with a spherical pendulum in the neighbourhood of internal and external resonance are investigated. It was assumed that the spherical pendulum is attached to the main body which is suspended to the element characterized by the elasticity and damping and it is excited harmonically in the vertical direction. The equations of the motion of this system were solved using numerical methods. In this type of the system, one mode of vibration can excite or damp another vibration. Or there can also appear different kinds of vibrations, in which we can find some chaotic vibrations.

The spherical pendulum is similar to the simple pendulum but it moves in three dimensional space. We introduced two models of this system (Fig 1).



a b

Fig. 1. Analysed models

The first model

First we assumed that the position of the main body is described by coordinate z and the position of the pendulum is described by the coordinate z and two angles : θ and φ (typical for spherical pendulum) (Fig 1a). Where the angle θ is the deflection of the pendulum measured from the vertical position. And angle φ describes the rotation of the pendulum in space xy.

The kinetic energy E_k and the potential energy E_p are given by the expression:

$$E_k = \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} = \frac{m_1 \dot{z}^2}{2} + \frac{m_2(\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2)}{2}$$

$$E_p = -(m_1 + m_2) g (z + z_{st}) + m_2 g (l - l\cos\theta) + \frac{k(z + z_{st})^2}{2}$$

where the Cartesian coordinates have the form:

$$
\begin{aligned}
x_2 &= l\sin\theta\cos\varphi \\
y_2 &= l\sin\theta\sin\varphi \\
z_2 &= z_1 + l\cos\theta \\
z_1 &= z + z_{st}
\end{aligned}
$$

Assuming that the exciting force is given in a form $F(t)=P\cos\nu_1 t$, the equations of the motion given as the Lagrange's equations are:

$$(m_1+m_2)\,\ddot{z} - m_2 l\ddot{\theta}\sin\theta - m_2 l\dot{\theta}^2\cos\theta + kz + c\dot{z} = P_1\cos\nu_1 t$$

$$m_2 l^2\ddot{\theta} - m_2 l\ddot{z}\sin\theta - m_2 l^2\dot{\phi}^2\sin\theta\cos\theta + m_2 gl\sin\theta = 0$$

$$m_2 l^2\ddot{\varphi}\sin^2\theta + 2m_2 l^2\dot{\varphi}\,\dot{\theta}\sin\theta\cos\theta = 0$$

Using Runge – Kutta method with a variable integration step, equations of the motion are solved numerically. Calculations have been done for different values of parameters and for different initial conditions. One of the example shows that there is the energy transfer between the modes of vibration (z and θ) in a closed cycle (angle φ is constant). That means that the spherical pendulum behaviors like the simple pendulum where the motion of pendulum is in the vertical plane. The influence of initial conditions on the autoparametric system with a spherical pendulum is very interesting. Sometimes, spherical pendulum is similar to the simple pendulum – when the initial conditions are put on the displacements. On the other hand, when the initial conditions are put on the velocities, there is an influence of the angle φ. It is very important due to the fact that near the internal and external resonance area different type of the motion either regular or chaotic can exist.

The second model

In order to check what type of the motion can be in our system with the spherical pendulum, we decided to change the variables in the system (Fig 1b). So, the position of the main body is described by coordinate z and position of the pendulum is describe by coordinate z and two angles: θ and φ in the vertical planes. Now we assume that there are generalized coordinates as follows: z, θ, φ. Coordinate z is the vertical displacement of the body of mass m_1 measured from the static position of equilibrium. The angle θ is the angle between the vertical ax and the deflection of the pendulum on the space xz . The angle φ is the angle between the deflections of the pendulum on the space xz and the pendulum The body of mass m_1 is subjected to the harmonic vertical excitation Coordinate z is the vertical displacement of the body of mass m_1 measured from the static position of equilibrium. The angle θ is the angle between the vertical axis and the deflection of the pendulum on the space xz . The angle φ is the angle between the deflections of the pendulum on the space xz and the pendulum The body of mass m_1 is subjected to the harmonic vertical excitation $F(t)=P_1\cos\nu_1 t$.
So in this case the Cartesian coordinates have the form:

$$
\begin{aligned}
x_2 &= l\cos\varphi\sin\theta \\
y_2 &= l\sin\varphi \\
z_2 &= l\cos\varphi\cos\theta + z_1 \\
z_1 &= z + z_{st}
\end{aligned}
$$

and the equations of the motion are now:

$$
\begin{aligned}
&(m_1+m_2)\,z \;\; - m_2 l\varphi\sin\varphi\cos\theta - m_2 l\varphi^2\cos\varphi\cos\theta - m_2 l\ddot{\theta}\cos\varphi\sin\theta + 2m_2 l\varphi\dot{\theta}\sin\varphi\sin\theta + \\
&\quad - m_2 l\dot{\theta}^2\cos\varphi\cos\theta + kz + c\dot{z} = P_1\cos\nu_1 t \\
&m_2 l^2\varphi - m_2 lz \;\; \sin\varphi\cos\theta + m_2 l^2\dot{\theta}^2\cos\varphi\sin\varphi + m_2 gl\sin\varphi\cos\theta = 0 \\
&m_2 l^2\ddot{\theta}\cos^2\varphi - 2m_2 l^2\dot{\theta}^2\cos\varphi\sin\varphi - m_2 lz \;\; \cos\varphi\cos\theta + m_2 gl\cos\varphi\sin\theta = 0
\end{aligned}
$$

Dynamics properties of the system are described by three differential equations containing strongly nonlinear terms which are investigated numerically. In this case we observe energy transfer between the all modes of vibration (z, θ, φ) in a closed cycle. In autoparametric system one mode of vibration may excite or damp another one, and for except periodic or quasi-periodic vibrations there may also appear chaotic vibration. For characterizing an irregular chaotic response, time histories, bifurcation diagrams, power spectral densities, Poincaré maps and maximal exponents of Lyapunov have been developed.
Model 2 gives much more interesting results then model 1.