A Multiple Scales Analysis of Very Large-Scale Arrays of Globally Coupled MEMS Resonators

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Summary. This work describes an analytical framework and multiple scales analysis of the dynamics of very large-scale arrays of globally coupled resonators, including parametric variability and nonlinearity. Most significantly, the analysis places no limits on the number of individual resonators that can be considered, extending to the infinite limit. Moreover, the resulting multiple scales analysis can predict the amplitude and phase dynamics of both the individual oscillators as well as the response of the overall population. This work is applied to a system of N non-identical resonators with global coupling, including both reactive and dissipative components, physically motivated by an electromagnetically-transduced microresonator array.

Discrete System of Resonators

We consider a generalization of an electromagnetically-transduced microresonator array that was explored, both analytically and experimentally, by Sabater et al. (2014) and (Sabater and Rhoads, 2015). This microsystem is actuated by Lorentz forces resulting from interactions between current-carrying conducting metal loops deposited on the resonator's surface and an external magnetic field. The global dissipative coupling arises naturally due to the current flowing through each individual resonator and the electromagnetic interactions described previously. This coupling is generalized to include a global reactive coupling component. Accordingly, the differential equations of motion that govern the coupled system's dynamics are given by

$$m_i\,\ddot{z}_i + c_i\,\dot{z}_i + k_i\,z_i + \gamma_i\,z_i^3 - \frac{\alpha}{N}\sum_{j=1}^{N}\dot{z}_j - \frac{\beta}{N}\sum_{j=1}^{N} z_j = f_i(t). \tag{1}$$

Here, each individual resonator in the N-element array is characterized by the index i and has an associated displacement z_i. Likewise, m_i, c_i, and k_i represent the mass, damping, and stiffness of each resonator, while γ_i parameterizes the strength of the nonlinearity. The parameters α and β characterize the overall strength of the dissipative and reactive global respectively. Finally, each resonator is subject to an external time-dependent excitation represented by $f_i(t)$. For N discrete oscillators a population density function $\rho_N(s)$ can be identified as

$$\rho_N(s) \equiv \frac{1}{N}\sum_{j=1}^{N}\delta(s-s_j). \tag{2}$$

The distribution parameter s is identified with each oscillator within the population. This system can therefore be nondimensionalized and scaled, so that Eq. (1) can be written as

$$\begin{aligned}(1+\epsilon\,\mu(s))\;\ddot{z}(t;s) + 2\,\epsilon\,\zeta(s)\,\dot{z}(t;s) + \left(\Omega^2+\epsilon\,\sigma(s)\right)\,z(t;s) + \epsilon\,\gamma(s)\,z^3(t;s)&\\ - \epsilon\,\alpha\int_{-\infty}^{\infty}\dot{z}(t;n)\,\rho(n)\,\mathrm{d}n - \epsilon\,\beta\int_{-\infty}^{\infty} z(t;n)\,\rho(n)\,\mathrm{d}n = \epsilon\,F_0\,\sin(\Omega\,t),&\end{aligned} \tag{3}$$

where $(\mu(s),\zeta(s),\sigma(s),\gamma(s))$ describe the appropriate distributions of mass, damping, linear stiffness, and nonlinearity. We note that in Eq. (3) the distribution parameter s identifies a particular resonator, but the dynamics of each resonator can be described by a single-degree-of-freedom, Duffing-like model, subject to the time-varying global coupling functions parameterized by α and β.

Continuum Multiple Scales

Therefore, a multiple scales approach is applied to each resonator in Eq. (3), so that

$$z(t;s) = z_0(t;s) + \epsilon\,z_1(t;s) + \mathcal{O}(\epsilon^2); \quad \text{with} \quad \frac{\mathrm{d}}{\mathrm{d}t} = \Omega\,\frac{\mathrm{d}}{\mathrm{d}\tau} + \epsilon\,\frac{\mathrm{d}}{\mathrm{d}\eta}. \tag{4}$$

To lowest order, the response of $z_0(t;s)$ is

$$z_0(t;s) = A(\eta;s)\,\sin(\tau) + B(\eta;s)\,\cos(\tau), \tag{5}$$

where $A(\eta;s)$ and $B(\eta;s)$ describe the slow dynamics of each oscillator, equivalent to the amplitude and phase of each oscillator with respect to the excitation. Introducing this solution into the next order of the approximation leads to secular

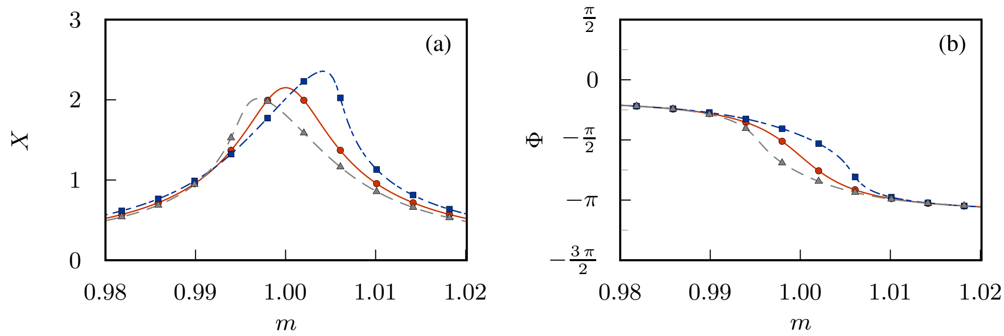

Figure 1: Nonlinear response functions, varying γ ($\alpha = 2.00$, $\beta = 5.00$, $N = 100$, iterative solutions); $\gamma = -0.10$, $\gamma = 0$, $\gamma = 0.10$; (a) X, (b) Φ.

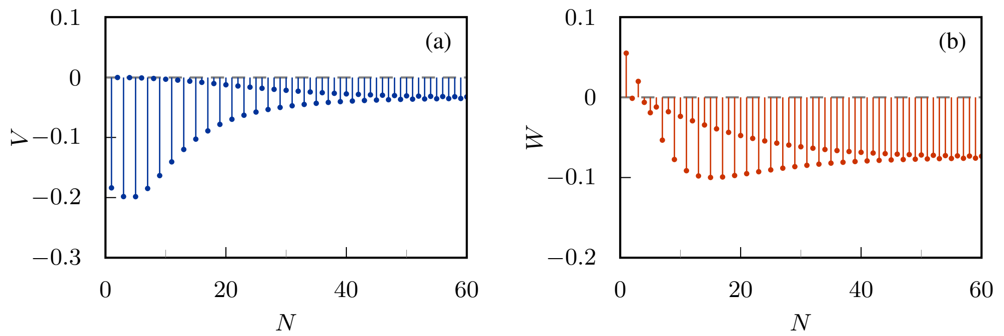

Figure 2: Global self-consistency constants, varying N ($\alpha = 2.00$, $\beta = 5.00$, $\gamma = 0$, iterative solutions); (a) V, (b) W.

terms that must be removed for a uniformly-valid solution. Specifically, the unspecified coefficients must satisfy

$$2\,\Omega\,\frac{\mathrm{d}B}{\mathrm{d}\eta} + F_0 + \big(\mu(s)\,\Omega^2 - \sigma(s)\big)\;A(s) + 2\,\zeta(s)\,\Omega\,B(s) - \frac{3\,\gamma(s)}{4}\,A(s)\;\big[A^2(s) + B^2(s)\big]$$
$$= \alpha\,\Omega\int_{-\infty}^{\infty} B(n)\,\rho(n)\,\mathrm{d}n - \beta\int_{-\infty}^{\infty} A(n)\,\rho(n)\,\mathrm{d}n, \qquad \text{(6a)}$$

$$2\,\Omega\,\frac{\mathrm{d}A}{\mathrm{d}\eta} + \big(-\mu(s)\,\Omega^2 + \sigma(s)\big)\;B(s) + 2\,\zeta(s)\,\Omega\,A(s) + \frac{3\,\gamma(s)}{4}\,B(s)\;\big[A^2(s) + B^2(s)\big]$$
$$= -\alpha\,\Omega\int_{-\infty}^{\infty} A(n)\,\rho(n)\,\mathrm{d}n - \beta\int_{-\infty}^{\infty} B(n)\,\rho(n)\,\mathrm{d}n. \qquad \text{(6b)}$$

These nonlinear integral equations can be solved for $A(s)$ and $B(s)$ to provide the $\mathcal{O}(\epsilon^0)$ approximation for the response of the population in the presence of global dissipative and reactive coupling. The integral terms are identified as the global self-consistency constants as

$$V \equiv \int_{-\infty}^{\infty} A(n)\,\rho(n)\,\mathrm{d}n, \quad W \equiv \int_{-\infty}^{\infty} B(n)\,\rho(n)\,\mathrm{d}n, \qquad (7)$$

and represent the influence of the global coupling on each resonator.

Analysis

These slow flow equations can be used to study the response of the system. For a system of N resonators in which only the mass m_i varies, the amplitude and phase response of the population is illustrated in Figure 1 as the stiffness nonlinearity γ varies. In addition, this analysis can be used to investigate how the response of the population varies as the number of resonators varies. In Figure 2 the global self-consistency constants are shown as N varies. Note that as N increases the values of (V, W) converge, so that the discrete population responds as a continuum.

References

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