Differential positivity for nonlinear consensus

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Summary. Linear positivity and its nonlinear generalization differential positivity provide both a novel interpretation and a suitable framework to study consensus. The agents in a linear consensus algorithm converge to consensus if and only if the linear dynamics is strictly positive (positivity is intended here with respect to general cones). Similar results hold for time varying and nonlinear consensus algorithms, through the novel approach of differential positivity.

Consensus algorithms are subject to widespread interest because of their relevance in distributed computation, control of collective behavior, coordination of networked systems and distributed sensing - to mention a few significant applications [2, 12, 4]. The collaborative effort of agents in consensus algorithms is typically organized around two main features: a shared objective (the invariant consensus manifold arising from the invariance properties of the interaction structure) and a distributed protocol to achieve that (a set of rules that drive the local behavior of the agents to asymptotically agree upon consensus) [11]. Convergence to consensus is thus regulated by communications and actuation constraints, which also limit performances and robustness of the algorithms [15, 13, 6]. The interaction among agents is usually represented by weighted graphs. Weights, typically positive, characterize the strength of the interaction, thus the speed of convergence towards consensus. The distinction between linear and nonlinear consensus follows from the linear/nonlinear nature of these interactions, [12, 14, 6, 3].

Linear consensus algorithms have a neat geometric interpretation in terms of invariant and contractive directions. Local rules act to reduce the disagreement among agents, that is, the distance from the consensus subspace (arrows in Figure 1). A consensus algorithm is thus contractive in directions transversal to the consensus manifold, as represented in the left part of Figure 1. The splitting between invariant (along the consensus manifold) and contractive directions (transversal to the consensus manifold) leads naturally to positivity, a system property characterized by the fact that the trajectories of the system leave a cone invariant (right part of Figure 1). In brief, given a pointed convex solid cone $\mathcal{K}$, a linear system $\dot{x} = Ax$ is positive if

$$e^{At}x \in \mathcal{K} \qquad \text{for all } x \in \mathcal{K} \text{ and all } t \geq 0$$

In this abstract we advocate that linear positivity [10] and its nonlinear generalization differential positivity [9] are strictly related to consensus algorithms and provide both a novel interpretation and a suitable framework to study consensus.

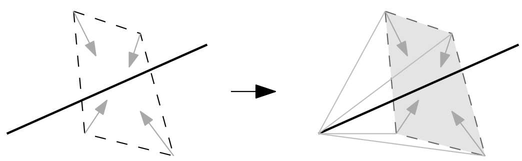

Figure 1: Consensus, projective contraction and strict positivity

Under mild conditions, positive systems enjoy a form of projective contraction, a feature relevant for consensus analysis. Perron-Frobenius theory guarantees the existence of an invariant subspace given by the span of a vector $v \in \mathcal{K}$ which is also an attractor for the system dynamics. Precisely, every ray of the cone $\lambda x \in \mathcal{K}$ asymptotically converges to the span of the dominant eigenvector, $\lim_{t\to\infty}\{\lambda e^{At}x \,|\, \lambda \geq 0\} = \{\lambda v \,|\, \lambda \geq 0\}$. In particular, contraction among the rays of the cone is guaranteed under *strict positivity* which further requires that, for some uniform interval $T > 0$,

$$e^{At}x \in \text{interior}\mathcal{K} \qquad \text{for all } x \in \mathcal{K} \setminus \{0\} \text{ and all } t \geq T$$

as shown in [5]. It is not hard to connect the projective contraction property of positive systems to the convergence of consensus algorithms. Because of the splitting, a linear consensus algorithm is a positive system: the consensus manifold identifies the attractor in $\mathcal{K}$. The existence of an invariant cone and convergence among the rays of the cone is guaranteed by the transversal contraction property of consensus algorithm. At the same time it is convenient to study strict positivity of a consensus algorithm since the agents of a strictly positive consensus algorithm asymptotically converge to consensus, from any initial condition: the contraction among the rays of the cone combined to the existence of an invariant direction naturally leads to contraction transversal to the consensus manifold, thus to the desired behavior at steady-state.

The connection between linear consensus and linear positivity is extensively studied in [16]. In the classical linear consensus algorithm the behavior of the ith agent is regulated by

$$\dot{x}_i = \sum_{j=1}^{n} \alpha_{ij}(x_j - x_i) \qquad \alpha_{ij} \geq 0\,.$$

In brief, the ith agent moves towards the weighted average given by $\sum_{j=1}^{n} \alpha_{ij} x_j$. The shape of the consensus manifold $\mathcal{A} := \{x \,|\, x_1 = \ldots = x_n\} = \{\lambda \mathbf{1} \,|\, \lambda \in \mathbb{R}, \mathbf{1} = (1, \ldots, 1)^T\}$ arises from the translation invariance of the interconnection structure, since $\dot{x}_i$ does not change if all agents x_j are translated to $x_j + \lambda$. Local contraction holds transversally to $\mathcal{A}$ since the overall system state matrix has $n-1$ stable eigenvalues. Figure 1 suggests that the system is positive. In fact, the consensus algorithm is a linear positive system with respect to the cone $\mathcal{K}$ given by the positive orthant $\mathbb{R}^n_+$. It is a standard exercise to show that strict positivity holds whenever the weights α_{ij} are strictly positive. Relaxed conditions can be provided.

Already in [16] the connection to positivity is exploited to characterize consensus algorithms in non-commutative spaces. The simplest example is given by the cone of positive definite matrices. In this abstract we simply observe that consensus algorithms with positive and negative weights (attractive and repulsive interactions among agents, [1])

$$\dot{x}_i = \sum_{j=1}^{N} \alpha_{ij}(x_j - x_i) - \sum_{j=1}^{N} \beta_{ij}(x_j - x_i) \qquad \alpha_{ij}, \beta_{ij} \geq 0$$

can be studied via linear positivity. The consensus manifold is still $\mathcal{A}$. If the overall system state matrix has $n-1$ stable eigenvalues then linear positivity must hold with respect to a cone $\mathcal{K} \neq \mathbb{R}^n_+$. Viceversa, establishing positivity with respect to some cone $\mathcal{K}$ guarantees asymptotic convergence of the consensus algorithm.

Time varying and nonlinear consensus algorithm calls for refined form of positivity analysis. Linear positivity is not enough. The analysis can be developed by looking at differential methods, in particular differential positivity and path-positivity [9, 7, 8]. In brief a nonlinear consensus $\dot{x} = f(x)$ can be studied by looking at its prolonged system

$$\dot{x} = f(x) \qquad \dot{\delta x} = \partial_x f(x) \delta x \ .$$

The system is strictly differentially positive with respect to the cone $\mathcal{K}$ if the trajectories of the prolonged system $(x(t), \delta x(t))$ satisfy, for some $T > 0$,

$$\delta x(t) \in \text{interior}\mathcal{K} \qquad \text{for all } \delta x(0) \in \mathcal{K} \setminus \{0\} \text{ and all } t \geq T \ .$$

Precise definitions, proper generalizations, connections to monotonicity and geometric conditions for positivity are extensively and rigorously discussed in [9, 7]. Within the scope of this contribution, we simply emphasize here that differential positivity is just positivity of the linearized dynamics. Its use for consensus analysis is analogous to the linear case.

The use of differential positivity for nonlinear consensus analysis is based on the following theorem: *consider a consensus algorithm $\dot{x} = f(x)$ strictly differentially positive system with respect to some cone $\mathcal{K}$. Suppose that $\mathbf{1}$ is an invariant direction of the linearization, that is, $\partial_x f(x)\mathbf{1} = 0$. Then, the trajectories of the consensus algorithm asymptotically converges to the consensus manifold $\mathcal{A}$.* The proof is a direct corollary of [7, Theorem 5]. An example (based on a slightly more general version of the theorem) is provided in [7].

In conclusion, positivity and differential positivity provide a natural tool for the analysis of linear and nonlinear consensus algorithms. The analysis of the convergence properties of consensus algorithms reduces to the study of the positivity of such systems.

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