An Approximation Method for Solving a Class of Time-delay Systems With Constant Time-delay

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Summary. This paper presents an approximation method to easily and effectively solve a class of delay differential equations (DDEs). The equation of motion of delayed system with constant delay is described in detail. The solution of the DDE consists of two parts, namely the solution of the open-loop system and that of the closed-loop system. The approximate solution of the DDE can expressed by finite poles and residues of the delayed system as well as other system parameters. Numerical simulations of both stable and unstable delayed systems are given to verify the proposed method.

Introduction

The effect of time-delay in dynamics and control has become a fundamental issue. Researches showed that the increase of time-delay from zero could cause the switch of stability [1]. The equation of motion of a time-delay system is a delay differential equation (DDE). Qin [2] once expanded the item with time-delay in DDE by Maclaurin series so that the DDE might be substituted by an ODE. However, results showed this expansion method would change the stability of the original delayed system, leading to the statement that DDEs cannot be expanded.

Description of the Time-delay Systems

The equation of motion of a time-delay system is a DDE. In this paper, we focus on a class of delayed systems with negative position feedback, which means the DDEs have the form as

$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=f(t)-gx(t-\tau) \quad t\in[0,t_d], \tag{1}$$

where g is the feedback gain and τ is the time-delay. The feedback parameters are positive. All the system parameters are constant. $f(t)$, the external excitation, can be any function with specific expression.
Initial conditions are given as follows.

$$x(0)=x_0,\ \dot{x}(0)=\dot{x}_0,\ x(t)=0 \ \ (-\tau\le t<0). \tag{2}$$

In this paper, we only discuss the systems where the eigenvalues are complex conjugate pairs.

$$s_i=\alpha_i+\mathrm{j}\beta_i,\ \ s_i^*=\alpha_i-\mathrm{j}\beta_i . \tag{3}$$

where β_i is the absolute value of the imaginary part. The real part can be negative or positive, which means the delayed system can be either stable or unstable.
The solution of Eq.(1) consists of two parts. The first part is the solution of an open-loop system over the time interval $t\in[0,\tau]$. We denote the displacement and velocity as x_{ol} and $\dot{x}_{ol}$, respectively. The second part is the solution of a closed-loop system over the time interval $t\in[\tau,t_d]$, denoted as $x_{cl}(t)$. The subscript 'cl' refers to closed-loop.

Closed-loop System

The equation of motion of the closed-loop system is a DDE, which can be expressed as

$$m\ddot{x}_{cl}(t)+c\dot{x}_{cl}(t)+kx_{cl}(t)=f(t)-gx_{cl}(t-\tau) \quad t\in[\tau,t_d]. \tag{4}$$

Initial conditions are denoted as x_{cl0} and $\dot{x}_{cl0}$, which are unknown and will be determined by later.
Taking the Laplace transform of Eq.(4) to get the solution of the closed-loop system in frequency domain.

$$X_{cl}(s)=(ms+c)x_{cl0}H_{cl}(s)+m\dot{x}_{cl0}H_{cl}(s)+F(s)H_{cl}(s), \tag{5}$$

where $X_{cl}(s)$ and $F(s)$ is the Laplace transform of the displacement and the external excitation, respectively. $H_{cl}(s)$ is the transfer function of the closed-loop system.

$$H_{cl}(s)=\frac{1}{ms^2+cs+k+ge^{-\tau s}} . \tag{6}$$

Taking the inverse Laplace transform of Eq.(5) to obtain the solution of the closed-loop system in time domain.

$$x_{cl}(t)=L^{-1}\left[(ms+c)x_{cl0}H_{cl}(s)+m\dot{x}_{cl0}H_{cl}(s)+F(s)H_{cl}(s)\right]. \tag{7}$$

Continuity conditions are given to determine the initial conditions of the closed-loop system.

$$\begin{cases} x_{cl}(\tau)=x_{ol}(\tau) \\ \dot{x}_{cl}(\tau)=\dot{x}_{ol}(\tau) \end{cases}. \tag{8}$$

If the values of x_{cl0} and $\dot{x}_{cl0}$ are obtained, then we can get the complete expression of the solution of the closed-loop system, which means the solution of the DDE (1) can be acquired.

Approximation of the Transfer Function

The theoretical transfer function (6) of the closed-loop system can also be expressed as

$$H_{cl}(s)=\sum_{i=1}^{\infty}\left(\frac{R_i}{s-s_i}+\frac{R_i^*}{s-s_i^*}\right), \tag{9}$$

where s_i is pole and R_i is residue. s_i^* and R_i^* represents their complex conjugate, respectively.

It can be noted that the theoretical transfer function is the sum of infinite items. However, it must be truncated and expressed by the sum of finite items due to feasibility. Thus, the transfer function of the closed-loop system can be approximated as follows.

$$H_{cl}(s)=\sum_{i=1}^{N}\left(\frac{R_i}{s-s_i}+\frac{R_i^*}{s-s_i^*}\right), \tag{10}$$

where N is the truncation order. s_i and R_i is pole and residue, respectively.

First, we sort the many eigenvalues according to the values of their real parts as

$$\alpha_1>\alpha_2>\cdots>\alpha_U>0>\alpha_{U+1}>\cdots>\alpha_N>\alpha_{N+1}>\cdots, \tag{11}$$

where U is the number of has unstable eigenvalues. Then the truncation order N is decided by the ratio of the real part of the first eigenvalue to the those of the other eigenvalues.

Simulation

Numerical solutions of DDEs can be obtained by using XPP-Aut and function dde23 in Matlab®. Since there is no analytical solutions of DDEs, these two solutions will be regarded as the 'exact solution', which the approximate solution will be compared with.

Stable system

The equation of motion of a stable delayed system is the same as (1). Assume that $m=1$ kg, $c=1$, $k=900$ N/m, $g=100$ N/m, $\tau=0.6$ s. The external excitation is set as the function of sine, namely $f(t)=100\sin(20t)$. Initial conditions are given as

$$x(0)=1,\ \dot{x}(0)=0.5,\ x(t)=0\ \ (-\tau\le t<0). \tag{12}$$

After using Vector Fitting [3] to fit the frequency response function (FRF) curve of the delayed system, we can obtain as many eigenvalues as we want. Choosing the truncation order N as 12 and using the approximation method, we can acquire the approximate solution of the stable delayed system.

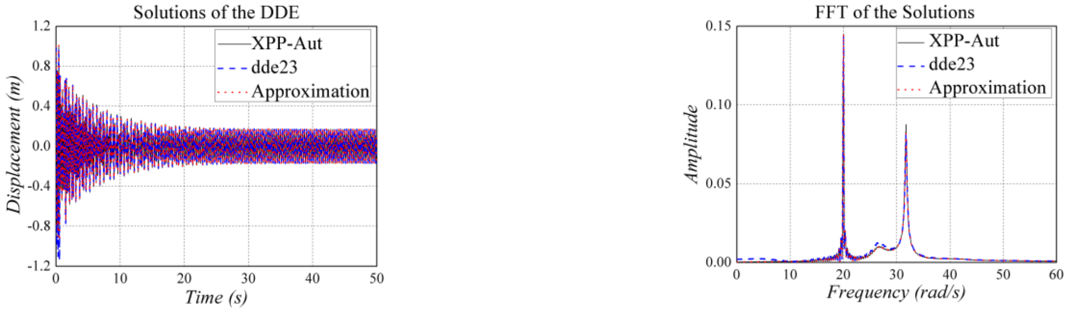

Fig. 1 the solutions of the DDE in time domain and frequency domain

Unstable system

The procedure in unstable delayed system is similar to that in stable delayed system.

Conclusion

After describing a class of delayed systems, we present an approximation method to solve these DDEs. The approximate solution of the DDE can expressed by finite poles and residues of the delayed system as well as other system parameters. Numerical simulations are given to verify the proposed method in both stable and unstable delayed systems.

References

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