Analysis of a remarkable singularity in a nonlinear DDE

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Summary. In this work we investigate the dynamics of the nonlinear DDE (delay-differential equation)

$$\frac{d^2x}{dt^2} + x(t-T) + x^3 = 0 \tag{1}$$

where T is the delay. In this system, the origin changes from stable to unstable as T is increased from 0. Associated with this transition is a remarkable bifurcation in which an infinite number of limit cycles exist for positive values of T in the neighborhood of $T = 0$, their amplitudes going to infinity in the limit as T approaches zero. We investigate this situation in three ways:
1) Harmonic Balance (HB),
2) Melnikov's integral,
3) Adding damping to regularize the singularity.

Harmonic Balance

Noting that the linear stability analysis of the system has been done by BHATT and HSU [1] we seek an approximate solution to eq.(1) in the form:

$$x(t) = A\cos\omega t \tag{2}$$

Substituting eq.(2) in eq.(1), simplifying the trig, and equating to zero the coefficients of $\sin\omega t$ and $\cos\omega t$ respectively, we obtain

$$\sin\omega T = 0 \qquad \text{and} \qquad -\omega^2 + \cos\omega T + \frac{3}{4}A^2 = 0 \tag{3}$$

The first of these gives $\omega T = n\pi$ for n=1,2,3,$\cdots$, whereupon the second gives

$$A = \frac{2}{\sqrt{3}}\sqrt{\frac{n^2\pi^2}{T^2} \pm 1}\ , \qquad n=1,2,3,\cdots \tag{4}$$

where the upper sign refers to n odd, and the lower sign refers to n even.

Melnikov's integral

We begin by writing eq.(1) as a perturbed conservative (Hamiltonian) system of the form:

$$\frac{dx}{dt} = \frac{\partial H}{\partial y} + g_1, \qquad \frac{dy}{dt} = -\frac{\partial H}{\partial x} + g_2 \tag{5}$$

$$\text{where} \quad H(x,y) = \frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{1}{4}x^4 \quad \text{and with perturbations} \quad g_1 = 0 \quad \text{and} \quad g_2 = x - x(t-T) \tag{6}$$

For the system (5), the condition for one of the closed curves $H(x,y) =$ constant to be preserved under the perturbation (5) turns out to be given by the vanishing of the following Melnikov integral:

$$\oint_\Gamma (g_1\dot{y} - g_2\dot{x})\,dt = \int_0^P -(x(t) - x(t-T))\dot{x}(t)dt = \int_0^P x(t-T))\dot{x}(t)dt = 0 \tag{7}$$

where Γ represents the unperturbed closed curve $H(x,y) =$ constant, where $\dot{x}$ and $\dot{y}$ refer to time histories around Γ in the unperturbed system, and where P is the period of the motion around Γ in the unperturbed system. The derivation uses Green's Theorem of the Plane, and the result is approximate (see section 3.3 in [2]). Here $x(t)$ is the solution to the unperturbed eqs.(5) with g_1=0 and g_2=0, which turns out to be a Jacobian elliptic cn function (see section 2.2 in [2]):

$$x = a_1\text{cn}(a_2 t, k) \quad \text{where} \quad a_2^2 = a_1^2 + 1, \qquad k^2 = \frac{a_1^2}{2(1+a_1^2)}. \tag{8}$$

Thus our Melnikov integral condition (7) simplifies to:

$$\int_0^P \text{cn}(a_2(t-T),k)\,\frac{d}{dt}(\text{cn}(a_2 t,k)\,dt = \int_0^P \text{cn}(a_2(t-T),k)\,\text{sn}(a_2 t,k)\,\text{dn}(a_2 t,k)\,dt = 0 \tag{9}$$

where $P = 4K(k)/a_2$, where $K(k)$ is a complete elliptic integral of the first kind. In order to obtain an analytical approximation for this integral, we approximate the elliptic functions sn, cn and dn by trig functions (see [3]) which gives

$$\sin(\pi a_2 T/(2K)) = 0 \;\Rightarrow\; \sin(\pi a_1 T/(2K(1/2)) = 0 \;\Rightarrow\; a_1 = 2Kn/T \;\Rightarrow\; a_1 \approx 3.71\, n/T. \tag{10}$$

where we have used the conditions $T << 1$ and $a_1 >> 1$, which from eqs.(8) gives $a_2 \approx a_1$, $k^2 = a_1^2/(2a_2^2) \approx 1/2$, and $K = K(1/2) \approx 1.854$. This result may be compared to the HB result of eq.(4), which is

$$a_1 \approx (2\pi/\sqrt{3})n/T \approx 3.63\, n/T. \tag{11}$$

Adding damping to regularize the singularity

We have seen that in the case of eq.(1), even infinitesimal delay gives rise to effective negative damping and growing oscillations. Accordingly, we expect that if damping is added to eq.(1), as in the case of the following DDE:

$$\frac{d^2x}{dt^2} + \alpha\frac{dx}{dt} + x(t-T) + x^3 = 0 \tag{12}$$

then if α is held fixed and delay T is increased from 0, there will be a point at which the equilibrium at the origin will make a transition from stable to unstable. Supposing that such a transition is a Hopf bifurcation, we linearize eq.(12) by dropping the x^3 term, and then set $x = \exp i\omega t$, giving the following real and imaginary parts, which when squared and added gives an expression for the critical delay for a Hopf bifurcation:

$$-\omega^2 + \cos\omega T = 0 \quad\text{and}\quad \alpha\omega - \sin\omega T = 0 \quad\Rightarrow\quad T_{crit} = \sqrt{2}\,\frac{\arccos\dfrac{-\alpha^2+\sqrt{\alpha^4+4}}{2}}{\sqrt{-\alpha^2+\sqrt{\alpha^4+4}}} \tag{13}$$

In addition to this Hopf bifurcation, it turns out that additional limit cycles can occur in this system by being born in a fold (also known as a saddle-node of cycles). In order to see this we again use the method of Harmonic Balance, which here gives (cf.eqs.(3)):

$$\sin\omega T = \alpha\omega \qquad\text{and}\qquad -\omega^2 + \cos\omega T + \frac{3}{4}A^2 = 0 \tag{14}$$

The first equation of (14) can be viewed in terms of two functions of the variable ω; one is the straight line $\alpha\omega$ of slope α and the second is the sinusoid $\sin\omega T$. See Fig.1.

As α is reduced, we go from curve a to b to c to d. Curve b corresponds to the Hopf bifurcation and the intersection

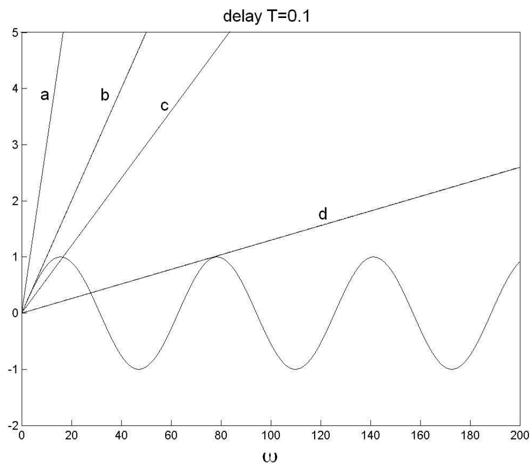

Figure 1: Graphical representation of the first of eqs.(14). The straight lines are $y = \alpha\omega$ and have slope α. The sinusoid is $y = \sin\omega T$.

of curve c with the sinusoid represents the limit cycle born in the Hopf bifurcation. The tangency between curve d and the sinusoid represents the saddle node bifurcation of cycles since further lowering α will turn the tangency into a pair of intersections, one corresponding to a stable limit cycle and the other to an unstable one. Once the frequency is specified, the amplitude of the motions gets determined by the second equation in (14).

References

[1] S.J.Bhatt and C.S.Hsu (1966) “Stability Criteria for Second-Order Dynamical Systems with Time Lag", Journal of Applied Mechanics 33(1): 113-118

[2] R.H.Rand (2012) “Lecture Notes in Nonlinear Vibrations" Published on-line by The Internet-First University Press http://ecommons.library.cornell.edu/handle/1813/28989

[3] P.F.Byrd and M.D.Friedman (1971) “Handbook of Elliptic Integrals for Engineers and Scientists", Second Edition, Springer-Verlag