Finding Periodic Solutions in the Dynamics of Metal Cutting via Averaging

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Summary. Regenerative machine tool vibrations are investigated in orthogonal cutting. Bifurcation analysis is presented for the governing nonlinear delay-differential equation, where the nonlinearity is represented by a Taylor series. Periodic solutions arising from Hopf bifurcation are calculated analytically using the method of averaging. Bistable technological parameter regions are determined by closed-form formulas.

Introduction

Machine tool vibrations produce noise, create poor surface finish, and limit the productivity of machining operations. Therefore, it is important to understand, suppress, and avoid machine tool chatter. One main source of machine tool vibrations is the surface regeneration effect, which can be described by delay-differential equations from dynamics point of view. The stability of their stationary solution determines the onset of chatter. Here, we investigate global stability properties and the phenomenon of bistability by analyzing the nonlinear dynamics of cutting.

We analyze the single-degree-of-freedom model of orthogonal cutting shown in Fig. 1a. Assuming a single dominant vibration mode, the tool's motion is governed by the following dimensionless equation

$$\ddot{x}(t) + 2\zeta\dot{x}(t) + x(t) = w \sum_{m=1}^{\infty} \eta_m \left(x(t-\tau) - x(t)\right)^m . \tag{1}$$

The damped oscillator on the left-hand side has a damping ratio ζ and is excited by the cutting force variation on the right-hand side. The cutting force variation is proportional to the dimensionless chip width w and is a function of the dimensionless chip thickness variation $x(t-\tau) - x(t)$, which is the difference of the tool's positions at the actual and the previous cut. The regenerative delay $\tau = 2\pi/\Omega$ is related to the angular velocity Ω of the workpiece. The cutting force variation is expanded into Taylor series with cutting-force coefficients η_m ($m \in \mathbb{Z}^+$), where $\eta_1 = 1$.

The trivial equilibrium of Eq. (1) represents stationary cutting, whereas its stability decides the onset of machine tool chatter. The linear stability boundaries of the equilibrium are well-known and read

$$w_{\rm st}(\omega) = \frac{\left(\omega^2-1\right)^2 + 4\zeta^2\omega^2}{2\left(\omega^2-1\right)} , \quad \Omega_{\rm st}(\omega) = \frac{\omega\pi}{j\pi - \arctan\left(\dfrac{\omega^2-1}{2\zeta\omega}\right)} , \quad j \in \mathbb{Z}^+ , \tag{2}$$

which can be depicted in the stability lobe diagram shown in Fig. 1b. The stability boundaries are associated with Hopf bifurcation, which is typically subcritical and gives rise to an unstable periodic solution with approximate angular frequency ω. The unstable solution makes the equilibrium's basin of attraction finite and affects global stability. As a result, there exists a region of bistability in the stability charts, where the linearly stable equilibrium coexists with the unstable periodic solution and is not stable in the global sense. Here, we determine the region of bistability by means of estimating the amplitude of the periodic solution analytically. Methods for computing periodic solutions include the center manifold reduction [1, 2] and the method of multiple scales [3, 4]. Here we use the method of averaging [5–9].

Method of averaging

We look for a harmonic approximation of the periodic solution in the form

$$x(t) \approx r(t)\cos(\omega t) , \quad \dot{x}(t) \approx -r(t)\omega\sin(\omega t) , \tag{3}$$

where the amplitude $r(t)$ is to be approximated by a constant. The amplitude $r(t)$ can be expressed as

$$r(t) = x(t)\cos(\omega t) - \dot{x}(t)\frac{1}{\omega}\sin(\omega t) . \tag{4}$$

Differentiating Eq. (4) and using Eqs. (1) and (3), we construct a differential equation for the amplitude $r(t)$. Then, we apply the theory of averaging by integrating from 0 to $2\pi/\omega$ and dividing by $2\pi/\omega$, which yields the average system

$$\dot{\overline{r}}(t) = F(\overline{r}(t)) . \tag{5}$$

Now we approximate the amplitude by a constant, $r(t) \approx \overline{r}(t) \approx \overline{r}_0$, which yields $\dot{\overline{r}}(t) \approx 0$. That is, we compute the equilibrium $\overline{r}_0$ of Eq. (5) satisfying $F(\overline{r}_0) = 0$, which implies

$$w_{\rm st} - w \sum_{k=1}^{\infty} \binom{2k-1}{k-1} \eta_{2k-1} \left(\overline{r}_0^2 \sin^2\left(\frac{\omega\tau_{\rm st}}{2}\right)\right)^{k-1} = 0 . \tag{6}$$

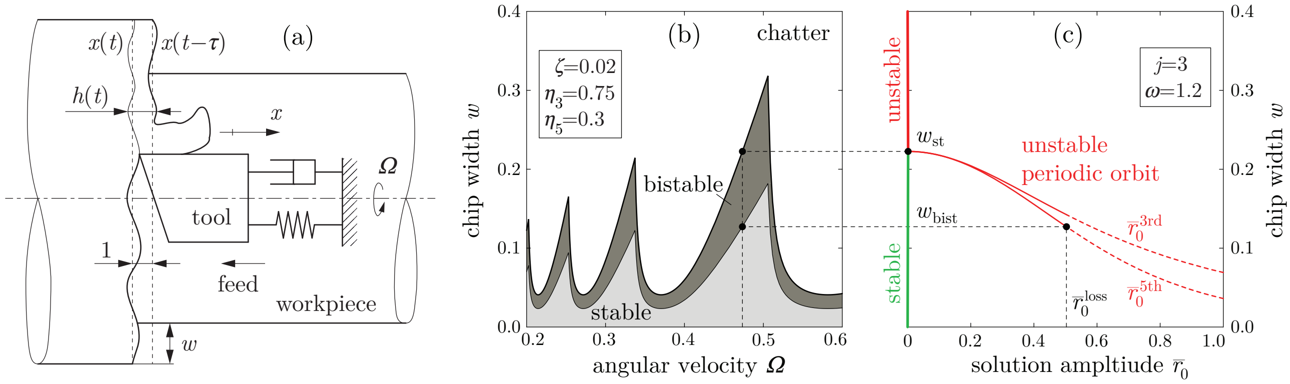

Figure 1: The mechanical model of orthogonal cutting (a); the corresponding stability lobe diagram (b); and bifurcation diagrams (c).

The algebraic equation (6) can be solved for the approximation $\overline{r}_0$ of the amplitude. In the case of cubic and quintic nonlinearities, we get the exact formulas

$$\overline{r}_0^{\mathrm{3rd}} = \sqrt{-\frac{w - w_{\mathrm{st}}}{3\eta_3 \sin^2\left(\frac{\omega\tau_{\mathrm{st}}}{2}\right) w}}, \quad \overline{r}_0^{\mathrm{5th}} = \sqrt{\frac{-3\eta_3 w + \sqrt{9\eta_3^2 w^2 - 40\eta_5 w (w - w_{\mathrm{st}})}}{20\eta_5 \sin^2\left(\frac{\omega\tau_{\mathrm{st}}}{2}\right) w}}. \tag{7}$$

The corresponding bifurcation diagrams are shown in Fig. 1c. The branches of periodic solutions are valid up to $\overline{r}_0 = \overline{r}_0^{\mathrm{loss}}$. At $\overline{r}_0 = \overline{r}_0^{\mathrm{loss}}$, the amplitude of the unstable periodic solution gets so large that the tool jumps out of the workpiece, loses contact during cutting, and the periodic solution vanishes. Thus, loss of contact sets the boundary w_{bist} of the bistable region. Loss of contact implies zero chip thickness, that is, $h(t) = 1 + x(t-\tau) - x(t) = 0$, which, combined with Eqs. (3) and (6), gives the following expression for the size of the bistable region

$$\frac{w_{\mathrm{st}} - w_{\mathrm{bist}}}{w_{\mathrm{st}}} = \frac{\sum_{k=2}^{\infty} \frac{1}{4^{k-1}} \binom{2k-1}{k} \eta_{2k-1}}{1 + \sum_{k=2}^{\infty} \frac{1}{4^{k-1}} \binom{2k-1}{k} \eta_{2k-1}} = \frac{\frac{3}{4}\eta_3 + \frac{10}{16}\eta_5 + \frac{35}{64}\eta_7 + \frac{126}{256}\eta_9 + \dots}{1 + \frac{3}{4}\eta_3 + \frac{10}{16}\eta_5 + \frac{35}{64}\eta_7 + \frac{126}{256}\eta_9 + \dots}. \tag{8}$$

Irrespective of the order of nonlinearity in the cutting force characteristics, formula (8) provides an easy way of estimating what percentage of the linearly stable region is bistable. This way, bistable parameter regions can be avoided and global stability is guaranteed for the cutting process. The approximate formula (8) has good accuracy when the periodic solution of the nonlinear differential equation is nearly harmonic.

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References

[1] Stépán G., Kalmár-Nagy T. (1997) Nonlinear regenerative machine tool vibrations. In *Proc of DETC'97, ASME Design and Technical Conferences*, 1-11, Sacramento, CA, USA.

[2] Dombóvári Z., Wilson R.E., Stépán G. (2008) Estimates of the bistable region in metal cutting. *P Roy Soc Lond A-Math Phy* **464**:3255-3271.

[3] Nayfeh A.H. (2008) Order reduction of retarded nonlinear systems – the method of multiple scales versus center-manifold reduction. *Nonlinear Dynam* **51**(4):483-500.

[4] Nandakumar K., Wahi P., Chatterjee A. (2010) Infinite dimensional slow modulations in a well known delayed model for cutting tool vibrations. *Nonlinear Dynam* **62**(4):705-716.

[5] Hale J., Verduyn Lunel S.M. (1990) Averaging in infinite dimensions. *J Integral Equations Appl* **2**(4):463-494.

[6] Lehman B., Weibel S. P. (1999) Fundamental theorems of averaging for functional differential equations. *J Differ Equations* **152**(1):160-190.

[7] Wahi P., Chatterjee A. (2004) Averaging oscillations with small fractional damping and delayed terms. *Nonlinear Dynam* **38**(1):3-22.

[8] Sanders J.A., Verhulst F., Murdock J. (2007) Averaging methods in nonlinear dynamical systems. Springer, NY.

[9] Sari T. (2007) Averaging for ordinary differential equations and functional differential equations. In van den Berg I., Neves V. (Eds.) *The Strength of Nonstandard Analysis*, 286-305, Springer, Wien.