Nonlinear model identification and spectral submanifolds for multi-degree-of-freedom mechanical vibrations

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Summary. In a nonlinear oscillatory system, spectral submanifolds (SSMs) are the smoothest invariant manifolds tangent to linear modal subspaces of an equilibrium. Amplitude-frequency plots of the dynamics on SSMs provide the classic backbone curves sought in experimental nonlinear model identification. We develop here a methodology to compute analytically both the shape of SSMs and their corresponding backbone curves from a data-assimilating model fitted to experimental vibration signals. Using examples of both synthetic and real experimental data, we demonstrate that this approach reproduces backbone curves with high accuracy.

Introduction

Finding nonlinear vibration modes is a long standing issue for mechanical systems. The aim is to find the closest possible analogue to linear vibration modes that can be used as a drop-in replacement. There are a number of competing definitions for nonlinear normal modes. The most general, evolved after Rosenberg's original definition [7], states that a nonlinear normal mode (NNM) is a family of periodic orbits of conservative systems. The definition by Shaw and Pierre, which is also applicable to damped vibrations, says that an NNM is an invariant manifold tangent to the subsapce of the corresponding linear mode. As it was pointed out by a number of authors this definition allows a multitude of wildly different invariant manifolds, making the definition useless. Haller [4] pointed out that, by effectively adding a smoothness criteria to the definition, the invariant manifolds can be made unique. Haller has called these unique manifolds spectral submanifolds (SSMs) to distiguish them from periodic or quasi-periodic orbits, which can also have SSMs themselves. The mathematical theory behind these definitions is due to Cabré et al.[1]. In what follows we use this theory for the case of sampled dynamics, which is directly compatible with experimental data collection techniques.
To utilise the theory, an underlying model of the mechanical system has to be constructed, which is addressed subsequently. Full description of the procedure can be found in [8].

Model identification

Our model identification technique is a straightforward application of Taken's delay embedding theorem to a nonlinear autoregressive (NAR) model. It is assumed that the vibration data is a scalar signal and sampled at regular time intervals with time period T. From this data a sequence of 2ν tuples are constructed, by selecting the past 2ν data points at every sample period. In essence, the vibration is embedded in $\mathbb{R}^{2\nu}$, where ν is sufficiently large to satisfy Taken's theorem [5]. For finding a single nonlinear mode of vibration $2\nu \geq 5$ is required. We write the resulting model as a map $\boldsymbol{x} \mapsto \boldsymbol{F}(\boldsymbol{x})$. Here, $\boldsymbol{F}$ is identified through the coefficients of its power series expansion.

SSMs and backbone curves

Consider a mechanical system with n degrees of freedom. For simplicity in this abstract it is assumed that all linear vibration modes are under damped. The natural frequencies are denoted by ω_j and the damping ratios are represented by $0 < \zeta_j < 1$. When selecting a natural frequency ω_ℓ and the corresponing linear subspace $\mathcal{E}_\ell$, the corresponding SSM is unique among the $C^{\sigma(\mathcal{E})+1}$ smooth invariant manifolds, where

$$\sigma(\mathcal{E}) = \operatorname{Int}\left[\frac{\max_{j\neq\ell,\ell+1} \zeta_j\omega_j}{\zeta_\ell\omega_\ell}\right],$$

In addition to smoothness there are finitely many non-resonance conditions, which are described in [8]. The beauty of the theory is that the vector field on the unique SSM is always polynomial of degree $\sigma(\mathcal{E}) + 1$.
The simplest non-trivial case occurs when $\sigma(\mathcal{E}) \leq 3$ and there is a single complex parameter β_ℓ that describes the nonlinear dynamics on the SSM. In this case the discrete time map $\boldsymbol{F}$ is reduced to

$$\begin{aligned}\rho_\ell &\mapsto \rho_\ell \left|\mu_\ell + \beta_\ell\rho_\ell^2\right|,\\ \theta_\ell &\mapsto \theta + \arg\left(\mu_\ell + \beta_\ell\rho_\ell^2\right),\end{aligned}$$

where ρ_ℓ is the amplitude and θ_ℓ is the phase parameter of the SSM. The oscillation frequency at a given amplitude is given by

$$\omega(\rho_\ell) = \frac{\arg\left(\mu_\ell + \beta_\ell\rho_\ell^2\right)}{T}. \tag{1}$$

Formula (1) can be used the recover a frequency-amplitude dependence on the SSM, which is also called the backbone curve [6]. This is an extension of the previous definition, because it can be used for damped oscillations. Formally, we then define the backbone curve as

$$\mathcal{B}_\ell = \{\omega(\rho_\ell), \mathrm{Amp}(\rho_\ell)\}_{\rho_\ell \in \mathbb{R}^+} \subset \mathbb{R}^2.$$

In a number of situations $\mathrm{Amp}(\rho_\ell) \approx \rho_\ell$, hence formula (1) can be used as a good approximation.

Experimental example

We consider the vibrations of clamped-clamped beam, whose nonlinearity is due to the boundary conditions. Vibrations of the beam is measured at its midpoint using a laser-doppler vibrometer, hence only velocity data is available. The details of the experiment and the data collected can be found in [2]. Here, the first three natural frequencies are investigated. Our model identification method also detects the natural frequencies and the damping ratios, which can be found in table 1.

Mode	$l = 1$	$l = 2$	$l = 3$
ω_l [Hz]	47.4921	167.1512	368.4577
ζ_l	0.1833	0.0183	0.0019
$\sigma(\mathcal{E}_l)$	0	2	12

Table 1: Natural frequencies and damping ratios for the first three modes of the clamped-clamped beam as determined by our algorithm. Also shown are the spectral quotients $\sigma(\mathcal{E}_l)$. The ω_l values are close to those linearly identified in [2], but the ζ_l values are markedly different.

The results of the backbone curve identification can be seen in figure 1. We used three free decaying vibration signals, all of which had initial conditions near the SSMs. These signals were incorporated into a single discrete time map $\boldsymbol{F}$ with $\nu = 3$. To calculate the SSMs we have used cubic polynomials, even though for the third natural frequency only a 13th order approximation would yield the unique expansion of the SSM. We note that the model identification technique fits the data globally, hence deviations in the natural frequencies are just as likely as for the nonlinear behaviour at higher amplitudes.

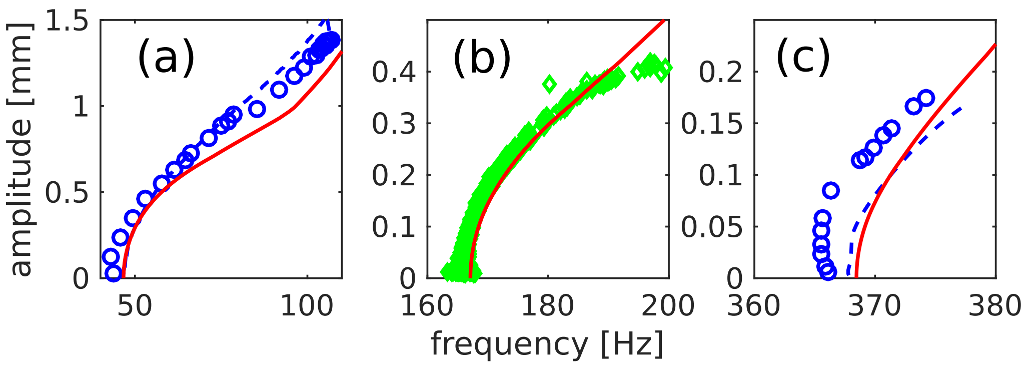

Figure 1: The first three backbone curves of a clamped-clamped beam. Red curves: backbone curves computed from a data-assimilating cubic-order SSM; Blue dashed lines: backbone curves obtained from individual decaying signals using a Hilbert transform approach [3]. Blue circles: force-appropriation results using stepped sine forcing. Green diamonds: Instantaneous amplitude-frequency curves inferred from decaying vibration data by calculating zero crossings of the signal to estimate vibration period. Apart from the red curves, all data was obtained directly from the experiments of Ehrhardt and Allen [2].

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