# Analysis of the Forced Vibration of Geometrically Nonlinear Cantilever Beam with Lumping Mass by Multiple-Scales Lindstedt-Poincaré Method

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*Summary*. The forced vibration of cantilever beam with or without lumping mass can be found in many practical applications. With the variational approach based on extended Hamiltonian principle, the equation of motion and boundary conditions governing a uniform cantilever beam carrying a lumping mass at the free end are formulated under the action of lateral tip concentrated load. After that, the multiple-scales method, multiple-scales Lindstedt-Poincaré method and fourth-order Runge-Kutta method are employed to analyze the forced vibration of the considered cantilever. It is found that the first mode frequency response curve obtained by the multiple-scales Lindstedt-Poincaré method agree well with the first mode frequency response curve obtained by the fourth-order Runge-Kutta method. However, the first mode frequency response curve obtained by multiple-scales method deviates from the first mode frequency response curve obtained by the fourth-order Runge-Kutta method when the beam deflection is large. When the excitation frequency is around second or third natural frequency, neither multiple-scales method nor multiple-scales Lindstedt-Poincaré method can give correct frequency response curves comparing to the frequency response curves obtained by the fourth-order Runge-Kutta method.

## Introduction

Perturbation theory consists of the methods for obtaining the approximate analytical solutions to nonlinea equations. For instance, algebraic equations, differential equations, integrals equations and integro-differential equations governing some physical systems can be approximately solved by perturbation methods. Some different perturbation methods such as multiple-scales (MS) method, Lindstedt-Poincaré method, multiple-scales Lindstedt-Poincaré (MSLP) method, the methods of matched and composite asymptotic expansion, and averaging method have been developed and employed [1, 2]. A small parameter must be introduced artificially to the equations when classic perturbation methods are employed. If the systems to be solved are strongly nonlinear, obtaining the approximate solutions to the problems becomes a challenge. The recent new theoretical results on the free and forced vibrations of some strongly nonlinear equations and oscillators are of great interest to the engineering community since their applications can be found in many areas. The new techniques can be summarized as 1) variational iteration method; 2) linearized perturbation method; 3) parameter expansion perturbation method; and 4) various modified Lindstedt-Poincaré methods. Each of these methods can be applied for obtaining the approximate solutions of a wide class of nonlinear systems without small perturbation parameter. Hu applied an iteration procedure for the solution of a quadratic nonlinear oscillator (QNO) and the obtained solution is improved in comparison with that obtained by the first-order harmonic balance method [4]. Marinca and Herisanu extended the iteration method and the obtained solution agrees well with exact solution [5]. Hu modified the equivalent linearization method for analyzing the nonlinear single-degree-of-freedom (SDOF) systems with odd nonlinearity [6]. A modified expansion method proposed by He achieved a high accuracy even when the perturbation parameter is within $0 \le \varepsilon < \infty$ [7]. Xu applied He's parameter-expanding method (PEM) to determine the limit cycles of some strongly nonlinear oscillators [8]. Comparing with the exact solution, PEM shows its effectiveness and accuracy for some nonlinear physical problems [9]. Cheung et al. introduced a new expanding parameter to Lindstedt-Poincaré method by which A strongly nonlinear system with large perturbation parameter is transformed into a system with small parameter [10]. A modified Lindstedt-Poincaré method was first proposed and the solution to a Duffing equation was obtained by Hu [11]. Hu and Xiong applied the modified Lindstedt-Poincaré method to a Duffing equation and compared the results with those from classical Lindstedt-Poincaré method [12]. They showed that the Lindstedt-Poincaré method is invalid when the perturbation parameter is large. The first-order and second order analytical approximate solutions with the modified Lindstedt-Poincaré method are formulated for a two-degree-of-freedom (TDOF) mass-spring system carrying quadratic nonlinearity by Lim et al [13]. An accurate result was achieved by them. In 2009, Pakdemirli proposed a method named multiple-scales Lindstedt-Poincaré method by combining multiple-scales method and Lindstedt-Poincaré method. This method has been applied to three oscillators, i.e., damped linear oscillator, Duffing oscillator, and damped Duffing oscillator. The results obtained by the MSLP method were compared with those from conventional MS method and numerical method. It is shown that the results from the MSLP method agree well with numerical simulation for some weakly and strongly nonlinear systems. Meanwhile, the results of conventional MS method deviate a lot from numerical simulation as the system nonlinearity increases [2]. Moreover, the MSLP method was applied to a damped duffing oscillator, a quintic duffing oscillator with strong nonlinearity and a quadratic nonlinear oscillator by Pakdemirli [14, 15, 16]. In this paper, the MS method, MSLP method and fourth-order Runge-Kutta method are employed to analyze the forced motion of the cantilever beam with lumping mass. The responses obtained by these methods are compared and the effectiveness of the MS and MSLP methods is examined by the results of numerical simulation.

# Vibrational analysis of the cantilever with large deformation

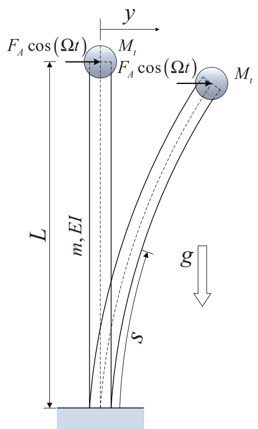



Figure 1: The scheme of the cantilever beam energy harvester with lumped mass on the free end.

Cantilever beam is widely used in many areas such as structural construction or energy harvester due to it can undertake strong forces and it can remain elastic even undergo a large deflection. However, only MS method has been adopted to analyzed the cantilever beams in the past [20, 21]. We consider the vibration of a cantilever with lumped tip mass and large deformation. An example is the energy harvester designed to be a cantilever with tip lumped mass as shown in Fig. 1. The cantilever is assumed to be an isotropic and inextensible Euler-Bernoulli beam. For flexible cantilevers, the large deformation can lead to obvious nonlinear behaviors. The MS method, with a small artificial parameter assumption, may be invalid in this case. In order to check the validity of each mentioned methods, the MS method, MSLP method and fourth-order Runge-Kutta method are applied to study the response of the cantilever undergoing planar motion. The analytical solutions obtained with the MS and MSLP methods are compared to the numerical solutions to figure out the effectiveness and accuracy of the MS and MSLP methods in various cases. The equation of motion of the cantilever with a lumped mass and tip harmonic excitation is given in the following [22].

$$
\begin{aligned}
&m\ddot{y}+M_t\delta(s-L)\ddot{y}(L,t)+c_y\dot{y}+EIy^{iv}=F_A\cos\left(\Omega t\right)\delta(s-L)-EI\left[y'(y'y'')'\right]' \\
&-\frac{1}{2}\left\{y'\int_L^s\left[\int_0^s y'^2 ds\right]^{\cdot\cdot} ds\right\}'+mg[(s-L)y''+y']+mg\left[(s-L)\frac{3y'^2y''}{2}+\frac{y'^3}{2}\right]
\end{aligned}
\tag{1}
$$

The boundary conditions are

$$
y(0,t)=0, y'(0,t)=0, y''(L,t)=0, y'''(L,t)=0 \tag{2}
$$

where $M_t$ is the lumped mass on the tip, $m$ is the beam mass per unit length, $L$ is the beam length, $E$ is Young's modulus, $I$ is the moment inertia of beam cross section, $g$ is the ground acceleration, $s$ is the arclength, $t$ is time, $y(s,t)$ is the transverse displacement, $c_v$ is the coefficient of linear viscous damping per unit length, $F_A$ is the forcing amplitude and $\Omega$ is the excitation frequency. The prime denotes the differentiation with respect to the arclength $s$. When the external excitation is nearby the $i$th natural frequency, the response is dominated by the $i$th mode. In this case, the approximate transverse displacement $y(s,t)$ of the beam is given by

$$
y(s,t)=\Phi_i(s)q_i(t) \tag{3}
$$

where $q_i(t)$ is the generalized coordinates corresponding to the $i^{th}$ mode, and $\Phi_i(s)$ is the $i^{th}$ linear mode function of the cantilever, which is expressed as

$$
\Phi_i(s)=\cos(p_is)-\cosh(p_is)-\frac{\cos(p_iL)+\cosh(p_iL)}{\sin(p_iL)+\sinh(p_iL)}\left[\sin(p_is)+\sinh(p_is)\right] \tag{4}
$$

With Galerkin's method and $y(s,t)=\Phi_i(s)q_i(t)$, the following system can be formulated.

$$
(1+\alpha\varepsilon)\ddot{q}_i+2u\varepsilon^2\dot{q}_i+\omega_0^2q_i+\beta\varepsilon q_i^3+\alpha\varepsilon q_i\dot{q}_i{}^2=F\varepsilon^2\cos(\Omega t) \qquad (i=1,2,3) \tag{5}
$$

where $\varepsilon$ is perturbation parameter and

$$\begin{aligned}
u\varepsilon^2 &= \omega_0\xi, && (6)\\
\omega_0^2 &= \frac{EI}{m_0}\int_0^L \Phi_i\Phi_i^{iv}ds - \frac{mg}{m_0}\left[\int_0^L (s-L)\Phi_i\Phi_i'' ds + \int_0^L \Phi_i\Phi_i' ds\right], && (7)\\
\alpha\varepsilon &= \frac{m}{m_0}\int_0^L \Phi_i\Phi_i''\int_L^s\int_0^s \Phi_i'^2 dsdsds + \frac{m}{m_0}\int_0^L \Phi_i\Phi_i'\int_0^s \Phi_i'^2 dsds, && (8)\\
\beta\varepsilon &= \frac{EI}{m_0}\left(\int_0^L \Phi_i\Phi_i''^3 ds + 4\int_0^L \Phi_i\Phi_i'\Phi_i''\Phi_i''' ds + \int_0^L \Phi_i\Phi_i'^2\Phi_i^{iv} ds\right)\\
&\quad -\frac{mg}{m_0}\left[\frac{3}{2}\int_0^L (s-L)\Phi_i\Phi_i'^2\Phi_i'' ds + \frac{1}{2}\int_0^L \Phi_i\Phi_i'^3 ds\right], && (9)\\
F\varepsilon^2 &= \frac{F_A}{m_0}\Phi_i(L), && (10)\\
m_0 &= M_t\Phi_i^2(L) + m\int_0^L \Phi_i^2 ds. && (11)
\end{aligned}$$

Eq. (5) is treated by the MS method and the MSLP method in the following, respectively.

**Multiple-scales method**

With MS method, the oscillator response is expressed as

$$q = q_0(T_0,T_1,T_2) + \varepsilon q_1(T_0,T_1,T_2) + \varepsilon^2 q_2(T_0,T_1,T_2) + O(\varepsilon^3) \tag{12}$$

where $T_0$, $T_1$ and $T_2$ are the fast and slow time scales which are given by

$$T_0 = t,\ T_1 = \varepsilon t,\ T_2 = \varepsilon^2 t. \tag{13}$$

By chain rule, the operators of time derivatives are

$$\frac{d}{dt} = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \ldots, \tag{14}$$

$$\frac{d^2}{dt^2} = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2(D_1^2 + 2D_0D_2) + \ldots, \tag{15}$$

where $D_n = \partial/\partial T_n$ and $D_n^2 = \partial^2/\partial T_n^2$. Substituting Eqs. (12), (14) and (15) into Eq. (5) and setting the coefficients of $\varepsilon^m$ $(m = 0,1,2)$ to zero lead to the following equations.

$$\begin{aligned}
&O(\varepsilon^0):\ D_0^2(q_0) + \omega_0^2 q_0 = 0, && (16)\\
&O(\varepsilon^1):\ D_0^2(q_1) + \omega_0^2 q_1 = -2D_0D_1(q_0) - \beta q_0^3 - \alpha q_0[D_0(q_0)]^2 - \alpha q_0^2 D_0^2(q_0), && (17)\\
&O(\varepsilon^2):\ D_0^2(q_2) + \omega_0^2 q_2 = -2D_0D_1(q_1) - D_1^2(q_0) - 2D_0D_2(q_0) - 2uD_0(q_0)\\
&\quad -3\beta q_0^2 q_1 - \alpha q_1[D_0(q_0)]^2 - 2\alpha q_0 D_0(q_0)D_0(q_1) - 2\alpha q_0 D_0(q_0)D_1(q_0) && (18)\\
&\quad -\alpha q_0^2 D_0^2(q_1) - 2\alpha q_0^2 D_0D_1(q_0) - 2\alpha q_0 q_1 D_0^2(q_0) + F\cos(\Omega t).
\end{aligned}$$

The solution to the $O(\varepsilon^0)$ equation is

$$q_0 = Ae^{i\omega_0 T_0} + \bar{A}e^{-i\omega_0 T_0} \tag{19}$$

where $A$ is a function of time scales $T_1$ and $T_2$ which can be determined by omitting the secular terms in the higher-order equation.

Substituting Eq. (19) into the righthand side of the $O(\varepsilon^1)$ equation and eliminating the secular terms yield

$$2i\omega_0 D_1(A) + 3\beta A^2\bar{A} - 2\alpha\omega_0^2 A^2\bar{A} = 0. \tag{20}$$

Then the solution of the $O(\varepsilon^1)$ equation can be obtained to be

$$q_1 = \Lambda e^{3i\omega_0 T_0} + \bar{\Lambda}e^{-3i\omega_0 T_0} \tag{21}$$

where

$$\Lambda = \frac{\beta A^3}{8\omega_0^2} - \frac{\alpha A^3}{4}. \tag{22}$$

Substituting the expressions of $q_0$ and $q_1$ into the $O(\varepsilon^2)$ equation along with the assumption of $\Omega = \omega_0 + \varepsilon^2\sigma$ and eliminating the secular terms yield

$$D_2(A) = \frac{Fe^{i\sigma T_2}}{4i\omega_0} - uA - \frac{9\alpha^2\omega_0 A^3\bar{A}^2}{4i} + \frac{9\alpha\beta A^3\bar{A}^2}{4i\omega_0} + \frac{15\beta^2 A^3\bar{A}^2}{16i\omega_0^3} \tag{23}$$

Then the solution to the $O(\varepsilon^2)$ equation can be obtained to be

$$q_2 = \Gamma_1 e^{3i\omega_0 T_0} + \Gamma_2 e^{5i\omega_0 T_0} + \bar{\Gamma}_1 e^{-3i\omega_0 T_0} + \bar{\Gamma}_2 e^{-5i\omega_0 T_0}, \tag{24}$$

where

$$\Gamma_1 = \frac{9\alpha^2 A^4 \bar{A}}{16} - \frac{\alpha\beta A^4 \bar{A}}{8\omega_0^2} - \frac{21\beta^2 A^4 \bar{A}}{64\omega_0^4}, \tag{25}$$

$$\Gamma_2 = \frac{3\alpha^2 A^5}{16} - \frac{\alpha\beta A^5}{8\omega_0^2} + \frac{\beta^2 A^5}{64\omega_0^4}. \tag{26}$$

Express the time derivative of $A$ as

$$\frac{dA}{dt} = \varepsilon D_1(A) + \varepsilon^2 D_2(A) + O(\varepsilon^3) \tag{27}$$

and a polar form of $A$ is assumed to be

$$A = \frac{1}{2} a e^{ib}. \tag{28}$$

Substituting Eqs. (20), (23) and (28) into Eq. (27) and setting the real and imaginary parts equal zero, respectively, yield

$$\dot{a} = \frac{f\varepsilon^2}{2\omega_0} \sin\gamma - ua\varepsilon^2 \tag{29}$$

and

$$\dot{\gamma} = \Omega - \omega_0 + \varepsilon\left(\frac{a^2\alpha\omega_0}{4} - \frac{3a^2\beta}{8\omega_0}\right) + \varepsilon^2\left(\frac{F\cos\gamma}{2a\omega_0} - \frac{9a^4\alpha^2\omega_0}{64} + \frac{9a^4\alpha\beta}{64\omega_0} + \frac{15a^4\beta^2}{256\omega_0^3}\right). \tag{30}$$

In steady state, $\dot{a}$ and $\dot{\gamma}$ equal to zeros. The frequency response curve can be obtained by eliminating $\gamma$ and $\sigma$ in Eq. (30) and $\Omega = \omega_0 + \varepsilon^2\sigma$. The relation between the excitation frequency and steady-state response can be obtained to be

$$\begin{aligned}\Omega =& \omega_0 + \varepsilon\left(-\frac{a^2\alpha\omega_0}{4} + \frac{3a^2\beta}{8\omega_0}\right) + \varepsilon^2\left(\mp\frac{F}{2a\omega_0}\sqrt{1 - \frac{4\omega_0^2 u^2 a^2}{F^2}} + \frac{9a^4\alpha^2\omega_0}{64}\right. \\ &\left. - \frac{9a^4\alpha\beta}{64\omega_0} - \frac{15a^4\beta^2}{256\omega_0^3}\right).\end{aligned} \tag{31}$$

The approximate response of the oscillator is finally obtained to be

$$q = a\{\cos(\Omega t - \gamma) + X_1\cos[3(\Omega t - \gamma)] + X_2\cos[5(\Omega t - \gamma)]\}, \tag{32}$$

in which

$$X_1 = a^2\varepsilon\left(\frac{9a^2\alpha^2\varepsilon}{256} - \frac{\alpha}{16} - \frac{21a^2\beta^2\varepsilon}{1024\omega_0^4} + \frac{\beta}{32\omega_0^2} - \frac{a^2\alpha\beta\varepsilon}{128\omega_0^2}\right) \tag{33}$$

and

$$X_2 = a^4\varepsilon^2\left(\frac{3\alpha^2}{256} + \frac{\beta^2}{1024\omega_0^4} - \frac{\alpha\beta}{128\omega_0^2}\right). \tag{34}$$

**Multiple-scales Lindstedt-Poincaré method**

The forced vibration of the same oscillator is analyzed with MSLP method in the following. With the MSLP method, a dimensionless parameter $\tau$ for the time transformation $\tau = \omega t$ is introduced to oscillator (5) first, which leads to

$$\omega^2 y'' + 2u\omega\varepsilon^2 y' + \omega_0^2 y + \alpha\varepsilon\omega^2 y'^2 y + \alpha\varepsilon\omega^2 y^2 y'' + \beta\varepsilon y^3 = F\varepsilon^2\cos\left(\frac{\Omega}{\omega}t\right), \tag{35}$$

in which $y(\tau) = q(\omega t)$ and the prime stands for derivative with respect to the new time variable $\tau$. The fast and slow time scales with respect to $\tau$ are

$$T_0 = \tau,\ T_1 = \varepsilon\tau,\ T_2 = \varepsilon^2\tau. \tag{36}$$

By chain rule, the operators of time derivatives are given by

$$\frac{d}{d\tau} = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \dots, \tag{37}$$

$$\frac{d^2}{d\tau^2} = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 D_1^2 + 2\varepsilon^2 D_0 + D_2 + \dots, \tag{38}$$

where $D_n = \partial/\partial T_n$ and $D_n^2 = \partial^2/\partial T_n^2$. The approximate solution is assumed to be

$$y = y_0(T_0,T_1,T_2) + \varepsilon y_1(T_0,T_1,T_2) + \varepsilon^2 y_2(T_0,T_1,T_2) + O(\varepsilon^3). \tag{39}$$

The following expression is used in the following analysis [2]

$$\omega_0^2 = \omega^2 - \varepsilon\omega_1 - \varepsilon^2\omega_2. \tag{40}$$

Substituting Eqs. (37)-(40) into Eq. (35) and equating coefficients of $\varepsilon^m$, $(m = 0,1,2,$ to zero lead to the following equations.

$$O(\varepsilon^0): \ \omega^2 D_0^2(y_0) + \omega^2 y_0 = 0 \tag{41}$$

$$\begin{aligned} O(\varepsilon^1): \ &\omega^2 D_0^2(y_1) + \omega^2 y_1 = -2\omega^2 D_0 D_1(y_0) + \omega_1 y_0 - \beta y_0^3 - \alpha\omega^2 y_0 [D_0(y_0)]^2 \\ &- \alpha\omega^2 y_0^2 D_0^2(y_0) \end{aligned} \tag{42}$$

$$\begin{aligned} O(\varepsilon^2): \ &\omega^2 D_0^2(y_2) + \omega^2 y_2 = -2\omega^2 D_0 D_1(y_1) - \omega^2 D_1^2(y_0) - 2\omega^2 D_0 D_2(y_0) \\ &- 2u\omega D_0(y_0) + \omega_2 y_0 + \omega_1 y_1 - \alpha\omega^2 y_1 [D_0(y_0)]^2 - \alpha\omega^2 y_0^2 D_0^2(y_1) \\ &- 2\alpha\omega^2 y_0 D_0(y_0) D_0(y_1) - 2\alpha y_0 \omega^2 D_0(y_0) D_1(y_0) - 2\alpha\omega^2 y_0^2 D_0 D_1(y_0) \\ &- 2\alpha y_0 y_1 \omega^2 D_0^2(y_0) - 3\beta {y_0}^2 y_1 + F\cos\left(\frac{\Omega}{\omega}\tau\right) \end{aligned} \tag{43}$$

The solution to the $O(\varepsilon^0)$ equation is obtained as

$$y_0 = A(T_1,T_2)e^{iT_0} + \bar{A}(T_1,T_2)e^{-iT_0}. \tag{44}$$

Eq. (44) is substituted into the righthand side the $O(\varepsilon^1)$ equation and the secular term is eliminated by

$$-2\omega^2 i D_1(A) + \omega_1 A + 2\alpha\omega^2 A^2\bar{A} - 3\beta A^2\bar{A} = 0. \tag{45}$$

It is assumed that $D_1(A) = 0$ since $\omega_1$ is real [2]. The frequency parameter $\omega_1$ is then expressed as

$$\omega_1 = 3\beta A\bar{A} - 2\alpha\omega^2 A\bar{A}. \tag{46}$$

With the secular term eliminated, the solution to the $O(\varepsilon^1)$ equation can be obtained as

$$y_1 = \Lambda e^{3iT_0} + \bar{\Lambda} e^{-3iT_0}, \tag{47}$$

where $\Lambda$ is given by

$$\Lambda = \frac{\beta A^3}{8\omega^2} - \frac{\alpha A^3}{4} \tag{48}$$

For the solution to the $O(\varepsilon^2)$ equation, the excitation frequency is related to the transformed frequency by

$$\Omega = \omega(1 + \varepsilon^2\sigma). \tag{49}$$

Substituting Eqs. (44), (47) and (49) into the righthand side of the $O(\varepsilon^2)$ equation and eliminating the secular term, it gives

$$\begin{aligned} &A\omega_2 - D_1^2(A)\omega^2 - 2iD_2(A)\omega^2 - 4i\alpha A\omega^2\bar{A}D_1(A) \\ &- \frac{3A^3\alpha^2\bar{A}^2\omega^2}{2} - \frac{3A^3\bar{A}^2\beta^2}{8\omega^2} - 2iu\omega A + \frac{3A^3\alpha\bar{A}^2\beta}{2} + \frac{F}{2}e^{i\sigma T_2} = 0. \end{aligned} \tag{50}$$

$D_2(A)$ may not equal zero since $\omega_2$ would be complex. Therefore the possible choice is $\omega_2 = 0$. Substituting $A = \frac{1}{2}ae^{ib}$ into Eq. (50) along with $\gamma = \sigma T_2 - b$ and setting the real and imaginary parts equal zero, respectively, yield

$$D_2(a) = \frac{F}{2\omega^2}\sin\gamma - \frac{ua}{\omega} \tag{51}$$

and

$$D_2(\gamma) = \sigma - \frac{3\alpha^2 a^4}{64} + \frac{3\alpha\beta a^4}{64\omega^2} - \frac{3\beta^2 a^4}{256\omega^4} + \frac{F}{2a\omega^2}\cos(\gamma). \tag{52}$$

The solution to the $O(\varepsilon^2)$ equation is given as

$$y_2 = \Gamma_1 e^{3iT_0} + \Gamma_2 e^{5iT_0} + \overline{\Gamma_1} e^{-3iT_0} + \overline{\Gamma_2} e^{-5iT_0}, \tag{53}$$

where

$$\Gamma_1 = A^3\left(\frac{\alpha\omega_1}{32\omega^2} - \frac{\alpha\beta A\bar{A}}{2\omega^2} - \frac{\beta\omega_1}{64\omega^4} + \frac{3\beta A\bar{A}}{32\omega^4} + \frac{5\alpha^2 A\bar{A}}{8}\right) \tag{54}$$

and

$$\Gamma_2 = A^5\left(\frac{3\alpha^2}{16} - \frac{\alpha\beta}{8\omega^2} + \frac{\beta^2}{64\omega^4}\right). \tag{55}$$

In steady-state, $D_2(a) = 0$ and $D_2(\gamma) = 0$, which can make $\gamma$ eliminated in Eqs. (51) and (52). Then the following expression of $\Omega$ about the frequency response curve can be obtained with Eq. (49).

$$\Omega = \omega\left[1 + \varepsilon^2\left(\frac{3\alpha^2 a^4}{64} - \frac{3\alpha\beta a^4}{64\omega^2} + \frac{3\beta^2 a^4}{256\omega^4} \mp \frac{F}{2a\omega^2}\sqrt{1 - \frac{4u^2\omega^2 a^2}{F^2}}\right)\right] \tag{56}$$

where $\omega = \sqrt{\frac{4\omega_0^2 + 3\varepsilon\beta a^2}{4 + 2\varepsilon\alpha a^2}}$. The approximate response of the oscillator is finally obtained to be

$$y = a\left\{\cos\left(\frac{\Omega}{\omega}t - \gamma\right) + X_1'\cos\left[3\left(\frac{\Omega}{\omega}t - \gamma\right)\right] + X_2'\cos\left[5\left(\frac{\Omega}{\omega}t - \gamma\right)\right]\right\} \tag{57}$$

where

$$X_1' = a^2\varepsilon\left(\frac{\beta}{32\omega^2} - \frac{\alpha}{16} + \frac{\alpha\omega_1\varepsilon}{128\omega^2} - \frac{\beta\omega_1\varepsilon}{256\omega^4} - \frac{a^2\alpha\beta\varepsilon}{32\omega^2} + \frac{3a^2\beta^2\varepsilon}{512\omega^4} + \frac{5a^2\alpha^2\varepsilon}{128}\right) \tag{58}$$

and

$$X_2' = a^4\varepsilon^2\left(\frac{3\alpha^2}{256} - \frac{\alpha\beta}{128\omega^2} + \frac{\beta^2}{1024\omega^4}\right). \tag{59}$$

## Comparison analysis

Eq. (5) is analyzed in the following by MS method, MSLP method and fourth-order Runge-Kutta method, respectively, to examine the effectiveness of MS and MSLP methods in different cases. The parameter values of a piezoelectric vibration energy harvester being a vertical cantilever with tip mass are taken from [23]. They are listed in Table 1. Corresponding

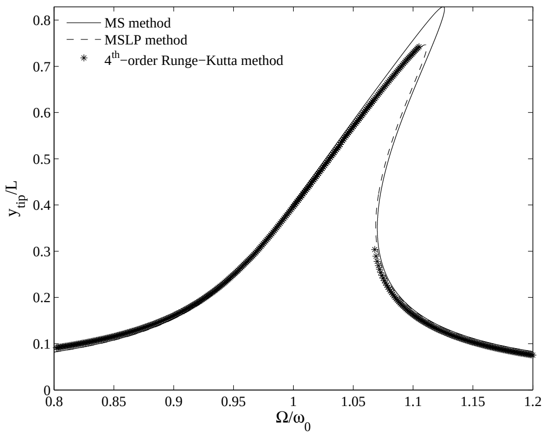
Table 1: Parameters of the cantilever energy harvester

| Symbol | Meaning | Value |
|---|---|---|
| $L$ | Length | 300 mm |
| $E$ | Young's modulus | 210 Gpa |
| $b$ | Width of cross section | 16 mm |
| $h$ | Height of cross section | 0.254 mm |
| $\xi$ | damping ratio | 0.02 |
| $I$ | Moment of inertia | $bh^3/12$ |
| $\rho$ | Material density | 7850 kg/m$^3$ |
| $m$ | Mass per unit length | $\rho A$ |
| $p_i$ | Characteristic value of beam | $1.8751/L$, $4.694/L$ or $7.855/L$ |
| $\omega_0$ | Natural frequency | $p_i^2\sqrt{EI/m}$ |
| $\mu$ | Tip and self mass ratio $M_t/mL$ | 0.5 |

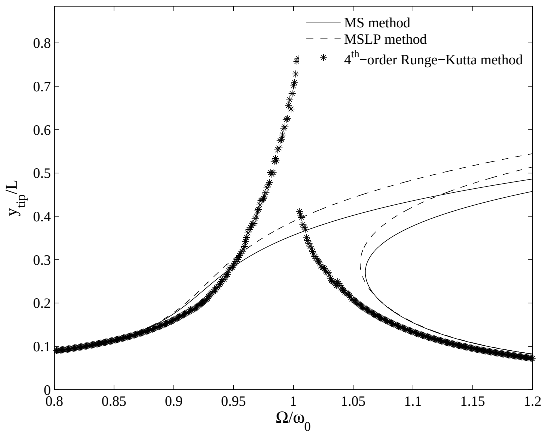
to each mode, the frequency response curves obtained by MS method, MSLP method and Runge-Kutta method are shown in Fig. 2. In the case of $i = 1$, the nonlinearity of restoring force dominates system nonlinearity when the excitation frequency is around the first natural frequency. In this case, the solution obtained by MSLP method agree well with the numerical solution while the results obtained by MS method deviate a lot from numerical simulation when the response amplitude is large as shown in Fig. 2(a). However, In the case of $i = 2$ or 3, the nonlinearity of inertial force dominates the system nonlinearity when the excitation frequency is around the second or third natural frequency. In this case, neither MS method nor MSLP method can give acceptable results in comparison to the results from numerical simulation as shown in Figs. 2(b) and 2(c).

## Conclusions

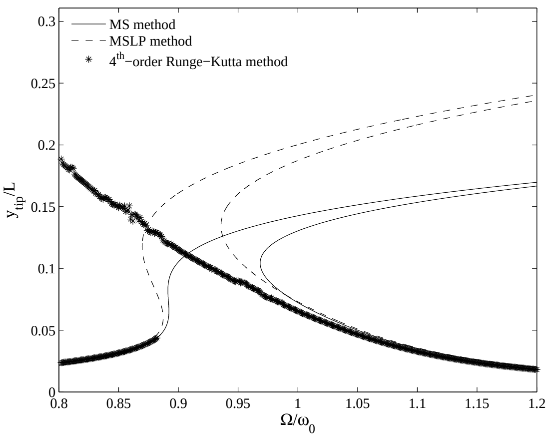
The responses of the forced vibration of the cantilever beam with lumping mass are investigated with MS method, MSLP method and numerical simulation, respectively, to examine the effectiveness of the MS method and MSLP method in various cases. The single mode frequency response curves of the cantilever obtained by MS method, MSLP method and




(a) Comparison of the first mode FRCs obtained by MS method, MSLP method and numerical simulation with parameters being $\omega_0 = 7.4853 rad/s$ and $F_A = 0.004N$.




(b) Comparison of the second mode FRCs obtained by MS method, MSLP method and numerical simulation with parameters being $\omega_0 = 52.7202 rad/s$ and $F = 0.2N$.




(c) Comparison of the third mode FRCs obtained by MS method, MSLP method and numerical simulation with forcing parameters being $\omega_0 = 149.1810 rad/s$ and $F = 0.4N$.

Figure 2: Comparison of the first three mode frequency response curves obtained by MS method, MSLP method and numerical simulations of a vertical cantilever beam energy harvester.

numerical simulation, respectively, are presented and compared to show the effectiveness of the MS method and MSLP method with different excitation frequencies. Generally speaking, for the forced vibrations of cantilever beams, nonlinear restoring force dominate the response of the beam when the excitation frequency is around first natural frequency. In the contrary, when the excitation frequency is around second, third or higher natural frequency, the nonlinear inertial force dominate the response of beam. Same phenomenon can be found that the first mode frequency response curve obtained by the multiple-scales Lindstedt-Poincaré method agree well with the first mode frequency response curve obtained by the numerical method. However, the first mode frequency response curve obtained by multiple-scales method deviates from the first mode frequency response curve obtained by numerical method when the beam deflection is large. When the excitation frequency is around second and third natural frequency, neither multiple-scales method nor multiple-scales Lindstedt-Poincaré method can give correct frequency response curves comparing to the frequency response curves obtained by the numerical method.

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