# Tracking critical points on evolving curves and surfaces

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Summary In recent years it became apparent that geophysical abrasion can be well characterized by the time evolution N(t) of the number N of static balance points of the abrading particle. Static balance points correspond to the critical points of the particle's surface represented as a scalar distance function r(u,v,t) measured from the center of mass of the particle. While N(t) is important for geophysicists, its computation poses challenges, because in the computational model r(u,v,t) is often replaced by its finely discretized approximation $r_\Delta(u,v,t)$ and the number $N_\Delta(t)$ of critical points corresponding to $r_\Delta(u,v,t)$ is, in general, not identical to N(t).We describe the geometric theory relating $N_\Delta(t)$ and N(t) and also provide an algorithm to compute N(t) based on $r_\Delta(u,v,t)$.

## Smooth curves, surfaces and their discretizations

We regard a smooth, closed, embedded convex curve C given as a scalar, polar distance r(φ), measured from the center of mass of the planar disc defined by C. We assume r(φ) to be a Morse function and call a point $r(\varphi_0)\in C$ a *static equilibrium point* if $r'(\varphi_0)=0$ (where ’ denotes dr/dφ). Depending on the sign of the second derivative we distinguish between stable and unstable equilibrium points and denote their numbers by S and U, respectively and as a trivial consequence of the Poincaré-Hopf Theorem [1] we have S=U. We refer to N=S+U as the number of *global equilibria* associated with C. In a numerical approximation C is often replaced by its fine polygonal discretization $c_\Delta$, which is obtained by constructing an equidistant Δ-mesh and connecting the meshpoints by straight lines. Analogously to the smooth curve, we may define the numbers $S_\Delta=U_\Delta$ associated with the polygon. In [2] we showed that in general, in the $\Delta \to 0$ limit $S_\Delta$ and $U_\Delta$ approach limit values $S_0>S$, $U_0>U$. We refer to $N_0=S_0+U_0$as the number of *local equilibria* associated with C. In [2] we gave explicit formulae to compute $N_0$, based on N, the location of the center of mass and the curvature of C. Local equilibria appear in spatially strongly localized “flocks” in the vicinity of global equilibria.

In 3 dimensions, the situation is analogous, however, here we have three types of equilibria and their respective numbers are related again by the Poincaré-Hopf Theorem: S+U-H=2 and in [2] we also provided the explicit formulae to compute the number of local equilibria $S_\Delta$, $U_\Delta$ and $H_\Delta$. If Δ is small but finite, we can visually observe the phenomenon: Figure 1 illustrates the flocks of local equilibria on finely discretized tri-axial ellipsoid.

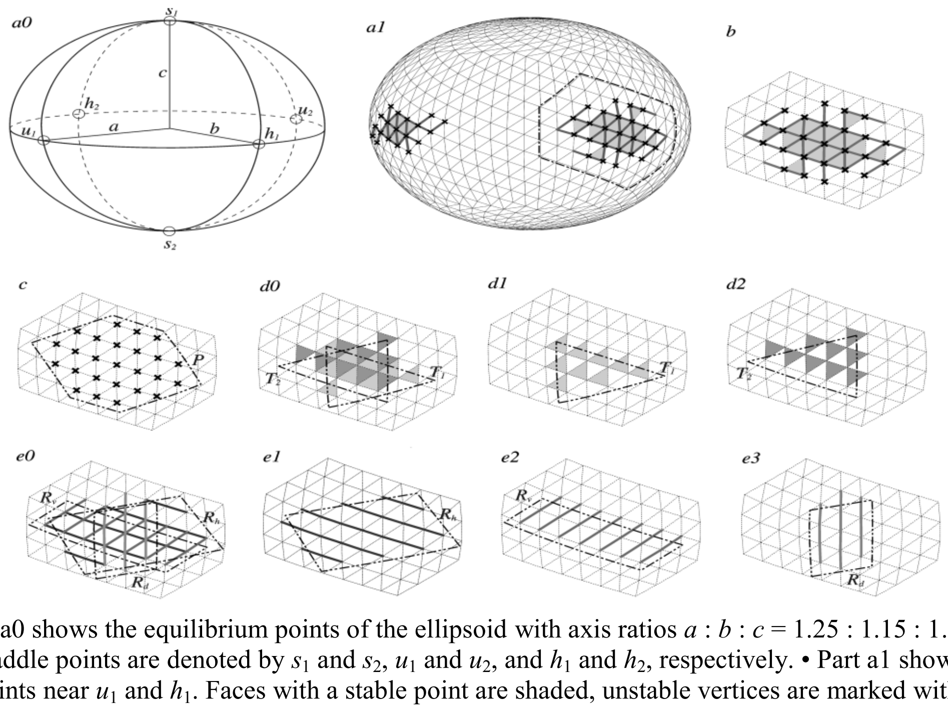


**Figure 1**: Part a0 shows the equilibrium points of the ellipsoid with axis ratios $a : b : c = 1.25 : 1.15 : 1$. The stable, unstable and saddle points are denoted by $s_1$ and $s_2$, $u_1$ and $u_2$, and $h_1$ and $h_2$, respectively. • Part a1 shows the equilibrium points near $u_1$ and $h_1$. Faces with a stable point are shaded, unstable vertices are marked with ×, and edges with a saddle point are drawn with bold lines. • Part b shows the equilibrium points near $h_1$. The zoomed hexagonal region is framed in Part a1. • Part c shows the unstable equilibrium points near $h_1$ inside the hexagonal region P. • Part d0 shows the stable equilibrium points near $h_1$ with the triangles T1 and T2. These triangles are separately shown in Parts d1 and d2, respectively. • Part e0 shows the saddle type equilibrium points near $h_1$ with the parallelograms R1, R2 and R3. Parts e1, e2 and e3 show these parallelograms separately.

## Co-evolution of local and global equilibria

In a discretized numerical scheme the surface C is represented by a set of points and local equilibria are the primary observable objects. In geometric evolution equations (such as curvature-driven flows [4]) the time evolution N(t) is often of prime interest, however, we can primarily observe $N_\Delta(t)$. Here we show that as a consequence of the results in [2], there is a remarkable coupling between the two functions: whenever N(t) suffers a jump, $N_0(t)$ escapes to infinity and, as a consequence, for sufficiently small $\Delta$, $N_\Delta(t)$ displays a sharp peak. Figure 2 illustrates this phenomenon on the time evolution of a planar curve under the curve-shortening flow [4].

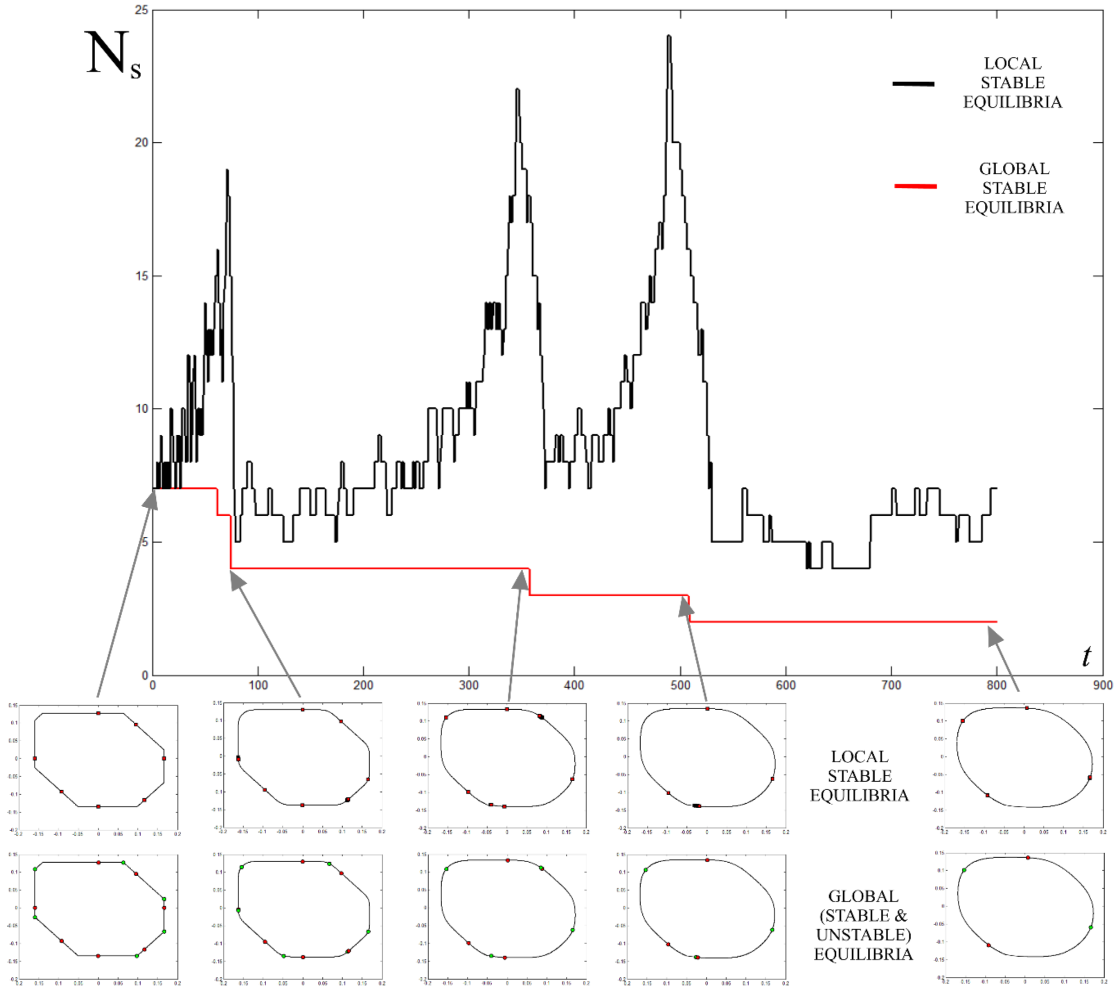


**Figure2**: Co-evolution of local and global equilibria under the curve-shortening flow. Contours are re-scaled to have constant area. Main plot: red lines shows N(t), black line shows $N_\Delta(t)$. Observe peaks of the latter coinciding with jumps of the former. Lower plots: distribution of equilibria in physical space.

## Conclusions

We showed how the number $N_0(t)$ of local equilibria on finely discretized curves and surfaces co-evolves with the number N(t) of global equilibria, associated with the smooth surface. The former is easier to observe, the latter is more important for physical applications so their co-evolution may offer a valuable tool for physical modeling.

## References

[1] Arnold, V.I., *Ordinary differential equations*, 10th printing, MIT Press, Cambridge, 1998.
*[2]* Domokos, G., Lángi, Z., & Szabó, T. (2012) On the equilibria of finely discretized curves and surfaces. *Monatshefte für Mathematik*, **168***(3-4), 321-345*. DOI: **10.1007/s00605-011-0361-x**
[3] Domokos, G., Lángi, Z., On global equilibria of finely discretized curves and surfaces. Arxiv preprint. arXiv:1410.5176 [math.DG]
[4] Domokos, G. (2015) Monotonicity of spatial critical points evolving under curvature-driven flows. *J. Nonlinear Sci.* **25**: 247. DOI:**10.1007/s00332-014-9228-3**