A Generalised Nonlinear Isolator-Elastic Beam Interaction Analysis for Extremely Low or High Supporting Frequency

Jing Tang Xing, Yeping Xiong, Kamal Djidjeli and Khairiah Kamilah Turahim
Faculty of Engineering & Environment, University of Southampton, Southampton, UK

Summary. This paper presents an interaction analysis of a generalised nonlinear isolator-elastic beam coupling system. In this system, an elastic beam-like structure is supported by a geometrically nonlinear suspension system of which a horizontal degree is introduced to provide a physical means for realizing the required horizontal force in some reported nonlinear isolation systems. The generalised dynamic equations of the interaction system are derived; in which three reduced models are obtained by introducing the related conditions into the generalised model. The nonlinear dynamic behavior on equilibria and stabilities of the system are investigated, and the dynamic interaction mechanism of the system is revealed. Following the mathematical analysis of the system, two examples illustrating applications of the developed theory are discussed. One simulates ground vibration tests of aircrafts, which requires an extreme low supporting frequency. Another involves structure dynamic tests in laboratories, where a rigid supporting foundation is expected. The investigated system can provide extremely low or high supporting stiffness and frequencies to satisfy special requirements for high precision vibration isolations.

Introduction

High performance vibration suspension systems with a very low or a very high stiffness are widely required in engineering applications. For ground vibration tests (GVT) of full scale aircrafts, the suspension frequency of the assumed rigid aircraft on the supporting system must be lower than one third of its first elastic natural frequency for accurate aircraft's flutter analysis. The weight of a large aircraft is very huge but its first elastic natural frequency is quite low so that the stiffness of the supporting system must have a big static stiffness to support the large weight and also a very low dynamic stiffness to have a very low supporting frequency [1-2]. The low supporting frequency is also a fundamental standard for effective vibration isolations [3-4] of high precision optical instruments used in space, such as for gravitational wave detection [5]. On the other hand, dynamic tests of structures in laboratories are often expected to be fixed on a rigid foundation, so that the stiffness of supporting system must be extremely high, otherwise, the foundation could not be considered as rigid. As we have learnt, an extremely "rigid" foundation for static tests could be very soft for dynamic tests with high frequencies. To design these supporting systems with particular performance, one approach is to adopt active feedback controls in a passive system [6-7]. This method requires an energy supply for the control system, which sometimes is difficult if the required energy is huge. Another one is to use nonlinear springs with a variable dynamic stiffness. For GVT of aircrafts, a nonlinear supporting system was proposed [1-2] to obtain a very low supporting frequency. The detailed investigations on designs, practical techniques and performance of nonlinear isolation systems were reported quite late, see, for example, references [8-14]. Moreover, nonlinear dynamical behavior on stabilities, bifurcations and chaos of this type of nonlinear isolation systems were not reported before recent publications [15-17]. While reading available publications on nonlinear isolation systems, we have noted that there are no publications involving integrated analysis on nonlinear isolator-structure interactions. As discussed for the analysis of structure-control interactions [18], the dynamic characteristics of both structures and control system are found to affect each other. Therefore, to assess the efficiency of a nonlinear isolation system, it is necessary to consider interactions. This paper intends to develop an integrated interaction analysis of a generalised nonlinear isolator-structures coupling system covering some reported systems by choosing its suitable parameters. This system introduces a horizontal degree physically to realise the required horizontal forces in [9] and [17]. Following a mathematical analysis of the system to develop the general theories on its equilibria, stabilities and small vibrations about each equilibrium point [19-21] and to reveal their interaction mechanism, two practical applications are studied. One simulates GVT of aircrafts to design a suspension system with a very low supporting frequency and another considers structure dynamic tests in laboratories, for which a rigid supporting foundation is expected.

Mathematical model of an integrated nonlinear isolator-beam interaction system

Fig. 1 shows the integrated system in which an elastic beam is supported by a generalised nonlinear suspension unit. The beam is uniform and subjected to two harmonic forces $F_0 \cos \Omega_0 t$ applied symmetrically at point ξ_0 in a coordinate system $O-\xi Y$ fixed at the middle point O of beam. There is a lumped mass $2M$ connected at point O by a rigid rod with its mass into $2M$. The beam is of span length $2S$, mass density ρ per unit length and bending stiffness $\Psi = EI$. The deflection $Y(\xi, t)$ of beam is a function of its material point ξ and time t. The mass $2M$ is supported by a generalised nonlinear suspension system symmetrical to the vertical axis $o-y$, and therefore it moves in the y direction only. The top ends of two linear inclined massless springs of stiffness k and non-stretched length l are connected to the mass $2M$ and their other two ends are respectively connected to two carts A and B of mass m allowing horizontal motions. There are two horizontal massless springs of stiffness K_1 with non-stretched length L_1 and two dampers of damping coefficient C_1 respectively connected to carts A and B positioning at x. Along the symmetrical axis $o-y$, a spring-damper set, consisting of a spring with stiffness $2K$ and its non-stretched length L with a damper of damping coefficient $2C$, is connected to mass $2M$. The model shown in Fig. 1 is a generalised model of structure-nonlinear isolator interaction system for practical designs in engineering. The horizontal spring-damper unit (K_1, C_1) aims to provide a means for the two horizontal forces added at the two carts in order to adjust the vertical dynamic stiffness of the total system. A suitable adjustment of the initial length L_1 of the spring K_1 results in a pull or push force applied at two carts A and B, which increases or decreases the vertical supporting stiffness of the system, respectively. The vertical spring-damper unit (K, C) supports the static weight of the mass $2M$ and the structure. The two spring-damper units (k, c) are the main elements with geometric nonlinear characteristics to adjust the dynamic supporting stiffness. The dampers in the system provide the adjusted parameters for the stability requirement of the system. Based on this generalised model, several simplified models [9, 15-17] can be obtained by introducing additional conditions or reducing some elements.

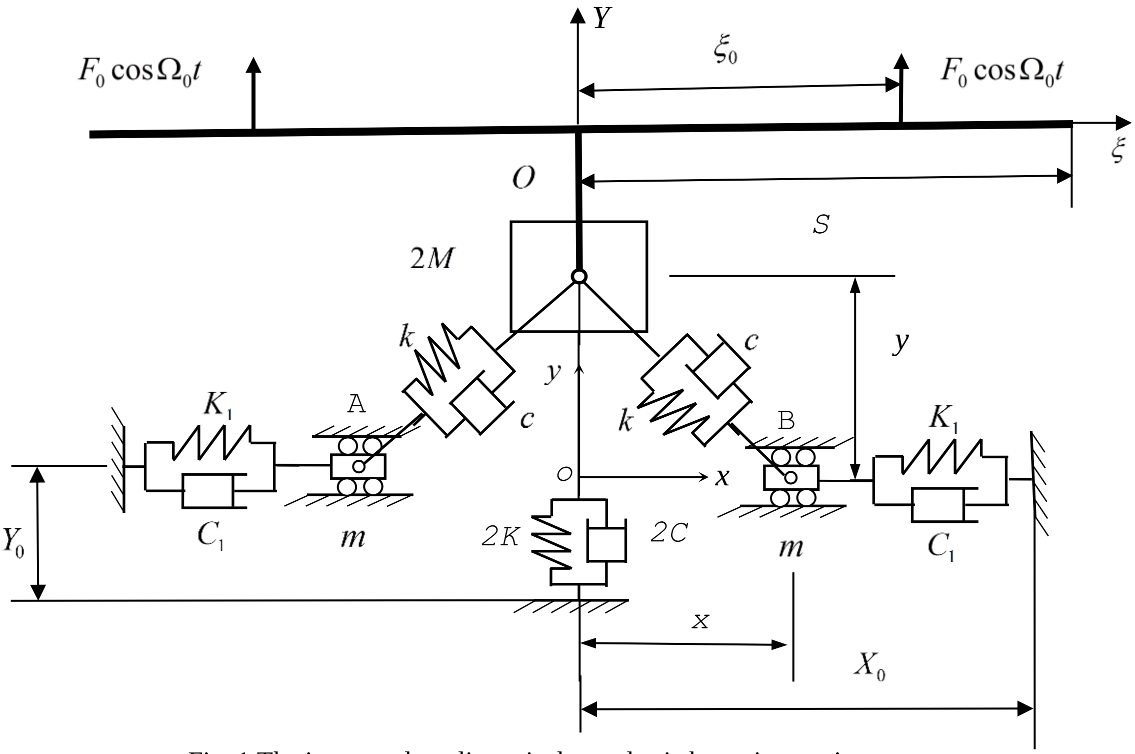

Fig. 1 The integrated nonlinear isolator-elastic beam interaction system

Governing equations of integrated interaction system

Considering the symmetry of system, we investigate its right half-part to derive the governing equations. It would be convenient to choose the origin o of coordinate system $o-xy$ and the origin O of beam coordinate system $O-\xi Y$ respectively located at their corresponding positions in a static equilibrium state, when the mass $2M$ and the two inclined springs k are on the horizontal axis $o-x$ with mass m at x_0. To realise this, we can choose a suitable extension $\Delta = Y_0 - L$ of the vertical spring K by investigating the static equilibrium of the system subject to the gravity only, i.e.

$$K\Delta = -g(M+\rho S), \qquad \Delta = -g(M+\rho S)/K, \qquad (K_1+k)x_0 - kl = K_1\Delta_1, \qquad \Delta_1 = X_0 - L_1. \tag{1.1}$$

Under these two reference coordinate systems, the gravity will be excluded in the governing equations and the variables y and $Y(\xi,t)$ represent the dynamic displacement of the mass $2M$ and the dynamic deflection of the beam relative to the static state, respectively. Obviously, at point $\xi = 0$, the function $Y(0,t) = y(t)$ is the vertical displacement of the lumped mass, due to rigid connection between them. In addition, the symmetrical conditions at point $\xi = 0$ of the beam require the shearing force and the rotation angle to be vanished at $\xi = 0$. Now, using the Newton's second law in association with the beam theory to investigate the dynamic equilibrium of beam, mass $2M$ and cart B, we derive the following governing equations.

Dynamic equilibrium equation and boundary conditions of beam structure

$$\Psi Y^{(\mathrm{IV})} + \rho\ddot{Y} = \delta(\xi-\xi_0)F_0\cos\Omega_0 t; \;\; Y'' = 0 = Y''', \;\; \xi = S; \;\; Y' = 0, \;\; \Psi Y''' = f_{bs}, \;\; \xi = 0. \tag{1.2}$$

Here, $()' = \partial()/\partial\xi$, $\dot{()} = \partial()/\partial t$, etc., f_{bs} represents a dynamic shearing force acting on the beam section $\xi = 0$ by the rigid rod, and $\delta()$ denotes the Delta function. The beam is a linear elastic structure, so that its motion can be represented using a mode superposition method [22]. In engineering, there are many nonlinear systems consisting of linear substructures connected by non-linear connectors, for which the mode superposition approach provides a very effective numerical model to formulate the motions of linear substructures [23]. The author has successfully and effectively used this approach to deal with numerical analysis for complex linear [18, 24-28] and nonlinear [29-32] dynamic systems involving fluid-structure interactions, control-vibration couplings, airplane landing impacts on VLFS, etc. This method is adopted to describe the beam motion. The deflection $Y(\xi,t)$ of the beam is represented by a mode summation form

$$Y(\xi,t)=\mathbf{Y}(\xi)\mathbf{\Phi}(t),\quad \mathbf{Y}=\begin{bmatrix}Y_1 & Y_2 & \cdots & Y_N\end{bmatrix},\quad \mathbf{\Phi}=\begin{bmatrix}\phi_1 & \phi_2 & \cdots & \phi_N\end{bmatrix}^T, \tag{1.3}$$
$$Y_n(\xi)=\frac{1}{2}\{\frac{\cosh(\lambda_n\xi/S)}{\cosh\lambda_n}+\frac{\cos(\lambda_n\xi/S)}{\cos\lambda_n}\},\qquad \tan\lambda_n+\tanh\lambda_n=0,\qquad n=1,2,3,\cdots.$$

based on the non-dimensional symmetrical mode functions $Y_n(\xi)$, $(n=1,2,...,N)$, of the uniform free-free beam. Here, N denotes a number of the retained mode functions $Y_n(\xi)$ and ϕ_n represents a generalised coordinate corresponding to mode n. These mode functions satisfy the following orthogonal relationships,

$$\int_0^S Y_n'' EIY_m'' d\xi=\begin{cases}0, & n\neq m,\\ K_{nn}, & n=m,\end{cases}\qquad \int_0^S Y_n\rho Y_m d\xi=\begin{cases}0, & n\neq m,\\ M_{nn}, & n=m,\end{cases}$$
$$M_{nn}=\begin{cases}\rho S, & n=1,\\ \rho S/4, & n\neq 1,\end{cases}\quad K_{nn}=\begin{cases}0, & n=1,\\ \dfrac{\lambda_n^4\Psi}{4S^3}, & n\neq 1,\end{cases}\quad \hat{\Omega}_n=\sqrt{K_{nn}/M_{nn}}=\frac{\lambda_n^2}{S^2}\sqrt{\frac{\Psi}{\rho}}. \tag{1.4}$$

The sub-index n indicates the mode number of the free-free beam, $\hat{\Omega}_n$, K_n and M_n represent the n-th natural frequency, generalised stiffness and mass, respectively. For the free-free beam, its first mode is a rigid mode with frequency $\hat{\Omega}_1=0$ and mode function $Y_1=1$. To distinguish this rigid mode with the elastic modes of the beam, a subscript "e" will be used to denote the related variables of the elastic modes, if this clarification is necessary in the following description. For examples, we can write $\mathbf{Y}=\begin{bmatrix}1 & \mathbf{Y}_e\end{bmatrix}$, $\mathbf{\Phi}=\begin{bmatrix}\phi_1 & \mathbf{\Phi}_e^T\end{bmatrix}^T$, etc. Substituting Eq.1.3 into Eq.1.2 and using the orthogonal relationships Eq.1.4, we obtain the following mode equation describing the beam motion

$$\mathbf{m}\ddot{\mathbf{\Phi}}+\mathbf{k}\mathbf{\Phi}=\mathbf{Y}^T(0)f_{bs}+\mathbf{Y}^T(\xi_0)F_0\cos\Omega_0 t,\qquad \mathbf{m}=\mathrm{diag}(M_{nn}),\qquad \mathbf{k}=\mathrm{diag}(K_{nn}),\qquad \mathbf{\Lambda}^2=\mathrm{diag}(\hat{\Omega}_n^2). \tag{1.5}$$

Dynamic equilibrium equations of the nonlinear supporting unit

$$\mathbf{M}\ddot{\mathbf{x}}+(\mathbf{C}+\mathbf{C}_c)\dot{\mathbf{x}}+(\mathbf{K}+\mathbf{K}_k)\mathbf{x}=\begin{bmatrix}K_1\Delta_1\\ f_{sb}\end{bmatrix}, \tag{1.6}$$

$$\mathbf{M}=\begin{bmatrix}m & 0\\ 0 & M\end{bmatrix},\ \mathbf{C}=\begin{bmatrix}C_1 & 0\\ 0 & C\end{bmatrix},\ \mathbf{C}_c=\frac{c}{\mu^2}\mathbf{x}\mathbf{x}^T,\ \mathbf{K}=\begin{bmatrix}K_1+k & 0\\ 0 & K+k\end{bmatrix},\ \mathbf{K}_k=-\frac{kl}{\mu}\mathbf{I},$$
$$\mathbf{x}=\begin{bmatrix}x & y\end{bmatrix}^T,\quad \Delta_1=X_0-L_1,\quad \mu=\sqrt{x^2+y^2}. \tag{1.7}$$

Here, Δ_1 represents the static extension of horizontal spring K_1 in the static state defined by Eq.1.1. The force f_{sb} denotes the reaction force from the beam to the lumped mass $2M$.

Interaction conditions between the beam structure and the nonlinear suspension unit

On the interaction section $\xi=0$ between the beam and the nonlinear suspension unit, a dynamic equilibrium condition and a geometrical constraint condition are required, i.e.

$$\text{Equilibrium:}\qquad f_{bs}+f_{sb}=0,\qquad -f_{bs}=f_{sb}=f, \tag{1.8}$$

$$\text{Geometrical constrain:}\qquad Y(0,t)=y(t). \tag{1.9}$$

By using Eq.1.3, this can be written in the mode form:

$$\mathbf{Y}_0\mathbf{\Phi}=y,\qquad \mathbf{Y}_0=\mathbf{Y}(0). \tag{1.10}$$

Non-dimensional dynamic equations

Equations 1.1-1.10 give the governing equations describing the dynamics of the integrated interaction system. To derive the non-dimensional equations of system, we introduce the following non-dimensional parameters,

$$\begin{aligned}
&\bar{x}=x/l,\ \ \bar{x}_0=x_0/l,\ \ \bar{y}=y/l,\ \ \bar{\Delta}_1=\Delta_1/l,\ \ \bar{\Delta}=\Delta/l,\ \ \bar{t}=\Omega_0 t,\ \ \bar{Y}=Y/l,\ \ \bar{\xi}=\xi/l,\ \ \bar{S}=S/l,\\
&\omega=\sqrt{(k+K_1)/m},\ \ \bar{\omega}=\omega/\Omega_0,\ \ E_1=C_1/(2m\omega),\ \ \bar{m}=m/M,\ \ \bar{\rho}=\rho l/M,\ \ \omega=\sqrt{\omega_k^2+\omega_1^2},\\
&\omega_k=\sqrt{k/m},\ \ \omega_1=\sqrt{K_1/m},\ \ \bar{\omega}_k=\omega_k/\Omega_0,\ \ \bar{\omega}_1=\omega_1/\Omega_0,\ \ \tilde{\omega}_k=\omega_k/\omega,\ \ \tilde{\omega}_1=\omega_1/\omega,\\
&\tilde{K}_1=K_1/(k+K_1),\ \ \bar{K}_1=\bar{m}\tilde{K}_1\bar{\omega}^2=K_1/(M\Omega_0^2),\ \ \bar{k}=\tilde{k}_K\bar{\Omega}^2=\bar{m}\tilde{k}_1\bar{\omega}^2=k/(M\Omega_0^2),\\
&\tilde{k}_1=k/(k+K_1),\ \ \tilde{k}_K=k/(k+K),\ \ \tilde{c}_1=c/C_1,\ \ \tilde{c}_K=c/C,\ \ \bar{\varepsilon}=\tilde{c}_K\mathrm{E}\bar{\Omega}=\bar{m}\tilde{c}_1E_1\bar{\omega}=c/(2M\Omega_0),\\
&\Omega=\sqrt{(k+K)/M},\ \ \bar{\Omega}=\Omega/\Omega_0,\ \ \mathrm{E}=C/(2M\Omega),\ \ \bar{g}=g/(\Omega_0^2 l),\ \ \Omega=\sqrt{\Omega_k^2+\Omega_K^2},\\
&\Omega_k=\sqrt{k/M},\ \ \Omega_K=\sqrt{K/M},\ \ \bar{\Omega}_k=\Omega_k/\Omega_0,\ \ \bar{\Omega}_K=\Omega_K/\Omega_0,\ \ \tilde{\Omega}_k=\Omega_k/\Omega,\ \ \tilde{\Omega}_K=\Omega_K/\Omega,\\
&\bar{F}_0=F_0/(M\Omega_0^2 l),\ \ \bar{f}_1=\bar{K}_1\bar{\Delta}_1,\ \ \bar{f}=f_{sb}/(M\Omega_0^2 l),\ \ \bar{\mathbf{m}}=\mathbf{m}/M,\ \ \bar{\mathbf{\Phi}}=\mathbf{\Phi}/l,\ \ \bar{\mathbf{\Lambda}}^2=\mathbf{\Lambda}^2/\Omega_0^2.
\end{aligned} \tag{1.11}$$

Dynamic equilibrium equation and boundary conditions of beam structure

$$\overline{\mathbf{m}}\ddot{\mathbf{\Phi}}+\overline{\mathbf{m}}\mathbf{\Lambda}^2\overline{\mathbf{\Phi}}=\mathbf{R}\bar{\mathbf{f}}+\overline{\mathbf{F}}_0,\quad \mathbf{R}=\begin{bmatrix}\mathbf{0} & -\mathbf{Y}_0^T\end{bmatrix},\ \overline{\mathbf{F}}_0=\mathbf{Y}_F^T\overline{F}_0\cos\bar{t},\quad \mathbf{Y}_F=\mathbf{Y}(\xi_0),\quad \mathbf{Y}_0\overline{\mathbf{\Phi}}=\bar{y}. \tag{1.12}$$

Dynamic equilibrium equations of the nonlinear supporting unit

$$\overline{\mathbf{M}}\ddot{\mathbf{q}}+2[\overline{\mathbf{M}}\overline{\boldsymbol{\varpi}}\mathbf{E}+\boldsymbol{\varepsilon}(\mathbf{q})]\dot{\mathbf{q}}+[\overline{\mathbf{M}}\overline{\boldsymbol{\varpi}}^2+\mathbf{k}(\mathbf{q})+\mathbf{k}_1(\mathbf{q})]\mathbf{q}=\bar{\mathbf{f}}, \tag{1.13}$$

$$\begin{gathered}\overline{\mathbf{M}}=\begin{bmatrix}\overline{m} & 0\\ 0 & 1\end{bmatrix},\ \overline{\boldsymbol{\varpi}}=\begin{bmatrix}\overline{\varpi} & 0\\ 0 & \overline{\Omega}\end{bmatrix},\ \mathbf{E}=\begin{bmatrix}E_1 & 0\\ 0 & \mathrm{E}\end{bmatrix},\ \mathbf{I}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ \mathbf{I}_1=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\ \mathbf{q}=\begin{bmatrix}q_1\\ q_2\end{bmatrix}=\begin{bmatrix}\bar{x}\\ \bar{y}\end{bmatrix},\\ \boldsymbol{\varepsilon}(\mathbf{q})=\frac{\bar{\varepsilon}}{\bar{\mu}^2}\mathbf{q}\mathbf{q}^T,\ \mathbf{k}(\mathbf{q})=-\frac{\bar{k}}{\bar{\mu}}\mathbf{I},\ \bar{\mathbf{f}}=\begin{bmatrix}0\\ \bar{f}\end{bmatrix},\ \mathbf{k}_1(\mathbf{q})=-\frac{\bar{f}_1}{q_1}\mathbf{I}_1,\ \bar{\mu}=\sqrt{\mathbf{q}^T\mathbf{q}}.\end{gathered} \tag{1.14}$$

Here, $\boldsymbol{\varepsilon}(\mathbf{q})$ and $\mathbf{k}(\mathbf{q})$ represent the nonlinear damping matrix and the nonlinear stiffness matrix of the system, respectively.

Integrated coupling matrix equation

Combining Eqs. 1.12 and 1.13, we obtain the integrated coupling equation of system in the matrix from

$$\widehat{\mathbf{M}}\ddot{\mathbf{Q}}+(\widehat{\mathbf{C}}^L+\widehat{\mathbf{C}}^N)\dot{\mathbf{Q}}+(\widehat{\mathbf{K}}^L+\widehat{\mathbf{K}}^N)\mathbf{Q}=\widehat{\mathbf{F}}_0, \tag{1.15}$$

$$\begin{gathered}\mathbf{Q}=\begin{bmatrix}q_1\\ \overline{\mathbf{\Phi}}\end{bmatrix},\ \mathbf{q}=\mathbf{T}\mathbf{Q},\ \widehat{\mathbf{M}}=\begin{bmatrix}\overline{m} & \mathbf{0}^T\\ \mathbf{0} & \overline{\mathbf{Y}}+\overline{\mathbf{m}}\end{bmatrix},\ \mathbf{T}=\begin{bmatrix}1 & 0\\ \mathbf{0} & \mathbf{Y}_0\end{bmatrix},\ \widehat{\mathbf{F}}_0=\begin{bmatrix}0\\ \overline{\mathbf{F}}_0\end{bmatrix},\ \widehat{\mathbf{C}}^N=2\mathbf{T}^T\boldsymbol{\varepsilon}(\mathbf{q})\mathbf{T},\\ \widehat{\mathbf{C}}^L=2\mathbf{T}^T\overline{\mathbf{M}}\overline{\boldsymbol{\varpi}}\mathbf{E}\mathbf{T},\ \widehat{\mathbf{K}}^L=\{\mathrm{diag}(0,\overline{\mathbf{m}}\mathbf{\Lambda}^2)+\mathbf{T}^T\overline{\mathbf{M}}\overline{\boldsymbol{\varpi}}^2\mathbf{T}\},\ \widehat{\mathbf{K}}^N=\mathbf{T}^T[\mathbf{k}(\mathbf{q})+\mathbf{k}_1(\mathbf{q})]\mathbf{T}.\end{gathered} \tag{1.16}$$

The total degree of freedom of this system is $1+N$ where N is the mode number chosen to describe the beam motion. We can rewrite Eq.1.15 in the phase space form

$$\begin{cases}\dot{\mathbf{Q}}=\mathbf{P}\\ \dot{\mathbf{P}}=\widehat{\mathbf{M}}^{-1}\{\widehat{\mathbf{F}}_0-(\widehat{\mathbf{C}}^L+\widehat{\mathbf{C}}^N)\mathbf{P}-(\widehat{\mathbf{K}}^L+\widehat{\mathbf{K}}^N)\mathbf{Q}\}\end{cases}. \tag{1.17}$$

Here, super-indices "L" and "N" identify the linear and nonlinear parts of matrices, respectively. The coupling matrix Eq.1.15 or 1.17 describes the dynamics of integrated interaction system. Based on these, we can investigate the coupling mechanism between the elastic beam and the nonlinear isolator and energy flow characteristics of system [33-34].

Interaction analysis

Equilibrium points

The static equilibrium points $\mathbf{Q}_0=\begin{bmatrix}q_{10} & \overline{\mathbf{\Phi}}_0\end{bmatrix}^T$ and $\overline{\mathbf{\Phi}}_0=\begin{bmatrix}q_{20} & \mathbf{0}^T\end{bmatrix}^T$ of the system can be derived from Eq. 1.15 by vanishing velocity $\dot{\mathbf{Q}}$, accelerations $\ddot{\mathbf{Q}}$ and external force $\widehat{\mathbf{F}}_0$ of the system, that is

$$\begin{gathered}(\overline{m}\overline{\varpi}^2-\bar{k}/\bar{\mu}_0-\bar{f}_1/q_{10})q_{10}=0,\ \bar{\mu}_0=\sqrt{q_{10}^2+q_{20}^2},\ \eta_1=\cos\vartheta=q_{10}/\bar{\mu}_0,\ \eta_2=\sin\vartheta=q_{20}/\bar{\mu}_0,\\ \left\{\begin{bmatrix}\overline{M}_{11} & 0 & \cdots & 0\\ 0 & \overline{M}_{22} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \overline{M}_{NN}\end{bmatrix}\begin{bmatrix}0\\ \hat{\Omega}_2^2\\ \vdots\\ \hat{\Omega}_N^2\end{bmatrix}+(\overline{\Omega}^2-\bar{k}/\bar{\mu}_0)\begin{bmatrix}1 & Y_{20} & \cdots & Y_{N0}\\ Y_{20} & Y_{20}^2 & \cdots & Y_{20}Y_{N0}\\ \vdots & \vdots & \ddots & \vdots\\ Y_{N0} & Y_{N0}Y_{20} & \cdots & Y_{N0}^2\end{bmatrix}\right\}\begin{bmatrix}q_{20}\\ 0\\ \vdots\\ 0\end{bmatrix}=\mathbf{0}.\end{gathered} \tag{2.1}$$

Table 1. The equilibrium points $(\beta=1,2,3,4)$ of the system

β	Horizontal coordinate		Vertical coordinate		$\bar{\mu}_0^{(\beta)}$
	$q_{10}^{(\beta)}$	$\eta_1^{(\beta)}$	$q_{20}^{(\beta)}$	$\eta_2^{(\beta)}$	
1	$(\bar{f}_1+\bar{k})/(\overline{m}\overline{\varpi}^2)=\tilde{k}_1+\tilde{K}_1\overline{\Delta}_1$	$+1$	0	0	$q_{10}^{(1)}=\tilde{k}_1+\tilde{K}_1\overline{\Delta}_1$
2	$-(\bar{f}_1+\bar{k})/(\overline{m}\overline{\varpi}^2)=-(\tilde{k}_1+\tilde{K}_1\overline{\Delta}_1)$	-1	0	0	$-q_{10}^{(2)}=\tilde{k}_1+\tilde{K}_1\overline{\Delta}_1$
3	$\bar{f}_1/(\overline{m}\overline{\varpi}^2-\overline{\Omega}^2)$	$q_{10}^{(2)}/\bar{\mu}_0^{(2)}$	$+\sqrt{\bar{\mu}_0^{2(2)}-q_{10}^{2(2)}}$	$q_{20}^{(2)}/\bar{\mu}_0^{(2)}$	$\bar{k}/\overline{\Omega}^2=\tilde{k}_K$
4	$\bar{f}_1/(\overline{m}\overline{\varpi}^2-\overline{\Omega}^2)$	$q_{10}^{(3)}/\bar{\mu}_0^{(3)}$	$-\sqrt{\bar{\mu}_0^{2(3)}-q_{10}^{2(3)}}$	$q_{20}^{(3)}/\bar{\mu}_0^{(3)}$	$\bar{k}/\overline{\Omega}^2=\tilde{k}_K$

Here, $-\pi/2\le\vartheta\le\pi/2$ represents the angle $\angle oBM$ between *o-x* axis and the right inclined spring. A positive value $q_{20}>0$ or $\eta_2>0$ implies the equilibrium point locates on the positive *o-y* axis. This set of equations is nonlinear. In general, an iteration approach is required to obtain its solution. However, as given in Eq.1.1, we have chosen the origin of reference system $o-xy$ at its static equilibrium position with $q_{10}=\bar{x}_0$ and $q_{20}=0$, which gives the equilibrium point (1). For the case of $q_{20}\ne 0$, the second equation in Eq.2.1 requires $\overline{\Omega}^2-\bar{k}/\bar{\mu}_0=0$, which is then substituted into the first equation in Eq. 2.1 to give the values of q_{10}. Table 1 lists the equilibrium points of the system. Physically, at an equilibrium point (β), the mass M and m are located at point $q_{20}^{(\beta)}$ on the vertical axis and point $q_{10}^{(\beta)}$ on the horizontal axis, respectively, while the beam is in its static deformation state with its middle point O follows the mass M locating at a corresponding position. The value of (η_1, η_2) defines an equilibrium point as well as the corresponding parameters of the system. For example, $\eta_2 = 0$ gives points (1) and (2) while $\eta_2 \neq 0$ defines points (3) and (4). To derive the equations which reveal the influence of the beam motions on the nonlinear suspension unit or vice versa, we can separately investigate the governing equation of the nonlinear unit or the beam which incorporates the effect of the other side, i.e. the beam or the nonlinear unit. The interaction conditions in Eqs.1.8-1.10 provide a bridge to obtain these equations as follows.

Equation governing the influence of beam motions on nonlinear suspension unit

Added parameters: From Eq.1.12, it follows that

$$\overline{\mathbf{Y}}\overline{\mathbf{\Phi}} = \mathbf{Y}_0^T \bar{y}, \quad \bar{y}^2 = \overline{\mathbf{\Phi}}^T \overline{\mathbf{Y}}\overline{\mathbf{\Phi}} = \begin{cases} 0, & \bar{y} = 0 \\ > 0, & \bar{y} \neq 0 \end{cases}, \quad \overline{\mathbf{Y}} = \mathbf{Y}_0^T \mathbf{Y}_0 = \overline{\mathbf{Y}}^T. \tag{2.2}$$

Therefore, the symmetrical matrix $\overline{\mathbf{Y}}$ is definitely positive and its inverse matrix exists, so that from Eq.2.2 we obtain

$$\overline{\mathbf{\Phi}} = \overline{\mathbf{Y}}^{-1} \mathbf{Y}_0^T \bar{y}. \tag{2.3}$$

Substituting Eq. 2.3 into Eq. 1.12, we obtain

$$\overline{\mathbf{m}}\overline{\mathbf{Y}}^{-1}\mathbf{Y}_0^T \ddot{\bar{y}} + \overline{\mathbf{m}}\overline{\mathbf{\Lambda}}^2 \overline{\mathbf{Y}}^{-1}\mathbf{Y}_0^T \bar{y} = -\mathbf{Y}_0^T \bar{f} + \mathbf{Y}_F^T \bar{F}_0 \cos \bar{t}, \tag{2.4}$$

which, when pre-multiplied by $\mathbf{Y}_0$, gives

$$\mathbf{Y}_0 \overline{\mathbf{m}}\overline{\mathbf{Y}}^{-1}\mathbf{Y}_0^T \ddot{\bar{y}} + \mathbf{Y}_0 \overline{\mathbf{m}}\overline{\mathbf{\Lambda}}^2 \overline{\mathbf{Y}}^{-1}\mathbf{Y}_0^T \bar{y} = -\mathbf{Y}_0\mathbf{Y}_0^T \bar{f} + \mathbf{Y}_0\mathbf{Y}_F^T \bar{F}_0 \cos \bar{t}. \tag{2.5}$$

For the free-free beam, it is not possible that the all modes $Y_n(0) = 0$, $(n = 1,2,..., N)$, so that the real number

$$\alpha_N = \mathbf{Y}_0 \mathbf{Y}_0^T = \sum_{n=1}^{N} Y_n^2(0) \neq 0, \tag{2.6}$$

and therefore from Eq.2.5 we derive the interaction force $\bar{f}$

$$\bar{f} = -m_b \ddot{\bar{y}} - k_b \bar{y} + f_b \bar{F}_0 \cos \bar{t}, \qquad m_b = \mathbf{Y}_0 \overline{\mathbf{m}}\overline{\mathbf{Y}}^{-1}\mathbf{Y}_0^T / \alpha_N, \quad k_b = \mathbf{Y}_0 \overline{\mathbf{m}}\overline{\mathbf{\Lambda}}^2 \overline{\mathbf{Y}}^{-1}\mathbf{Y}_0^T / \alpha_N, \quad f_b = (\mathbf{Y}_0 \mathbf{Y}_F^T / \alpha_N). \tag{2.7}$$

Replacing the interaction force $\bar{f}$ in Eq. 1.13 by Eq.2.7, we derive

$$\begin{aligned} &(\overline{\mathbf{M}} + \mathbf{m}_b)\ddot{\mathbf{q}} + 2[\overline{\mathbf{M}}\overline{\boldsymbol{\omega}}\mathbf{E} + \boldsymbol{\varepsilon}(\mathbf{q})]\dot{\mathbf{q}} + \{\mathbf{k}_b + [\overline{\mathbf{M}}\overline{\boldsymbol{\omega}}^2 + \mathbf{k}(\mathbf{q}) + \mathbf{k}_1(\mathbf{q})]\}\mathbf{q} = \mathbf{f}_b, \\ &\mathbf{m}_b = \mathrm{diag}(0, m_b), \qquad \mathbf{k}_b = \mathrm{diag}(0, k_b), \qquad \mathbf{f}_b = [0 \quad f_b]^T \bar{F}_0 \cos \bar{t}. \end{aligned} \tag{2.8}$$

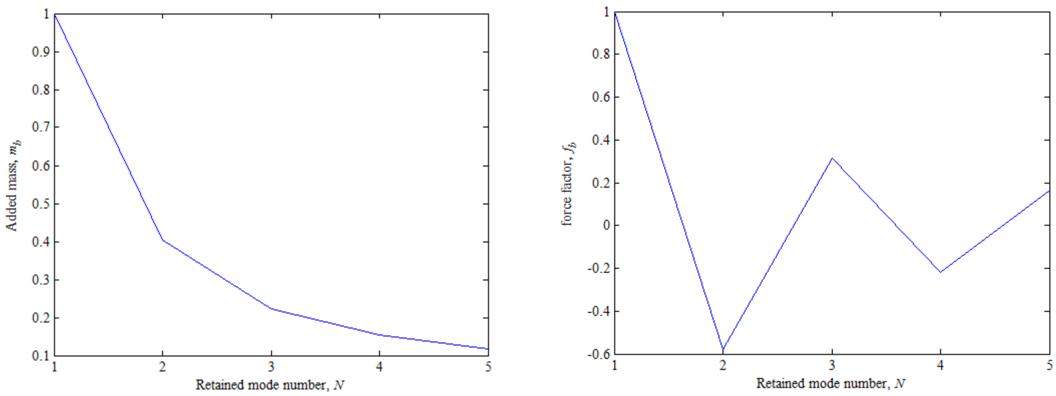

Fig. 2 the added mass and force factor affected by the retained modes

Here, m_b and k_b represent an additional dynamic mass and stiffness, respectively, which are added to the nonlinear suspension system by the beam due to their dynamic interactions. f_b defines a force factor at which the excitation force added to the lumped mass. Equation 2.8 provides a means to investigate the nonlinear support interacting with the elastic beam, when designing it to support a structure. The values of these added parameters depend on the retained mode number of the beam. For example, when only rigid mode $Y_1 = 1$ is retained, $N = 1$ in Eq.1.3 and therefore we have

$$\overline{\mathbf{m}} = \bar{m}_{11} = \bar{\rho}\bar{S}, \qquad \alpha_N = 1, \qquad m_b = \bar{\rho}\bar{S}, \qquad k_b = 0, \qquad f_b = 1. \tag{2.9}$$

Physically, m_b in Eq. 2.9 is the total mass of the beam. Since the beam is considered rigid and there is no elastic deformation, the added stiffness $k_b = 0$, and the force factor $f_b = 1$. Fig. 2 shows the added mass and force factor affected by the number $N = 1 \sim 5$ of retained modes.

Nonlinear stiffness force and potential energy: The two components of stiffness forces are

$$\mathbf{F}_R = \{\mathbf{k}_b + [\overline{\mathbf{M}}\overline{\boldsymbol{\omega}}^2 + \mathbf{k}(\mathbf{q}) + \mathbf{k}_1(\mathbf{q})]\}\mathbf{q}, \tag{2.10}$$

of which the horizontal component is not affected by the beam motion while the vertical component is affected by the additional added stiffness of the beam onto the nonlinear supporting unit. We choose the static position $\mathbf{q}_0 = [q_{10} \quad 0]^T$ which satisfies Eq.1.1 as a zero point of the potential energy, that is

$$\Pi[\mathbf{q}_0] = \bar{m}\bar{\varpi}^2 q_{10}^2 - \bar{k}q_{10} - \bar{f}_1 q_{10} = 0. \tag{2.11}$$

The potential energy at position $\mathbf{q}$ of the system is given by

$$\begin{aligned}\Pi[\mathbf{q}] &= \int_{\mathbf{q}_0}^{\mathbf{q}} d\mathbf{q}^T\mathbf{F}_R = [\frac{1}{2}\mathbf{q}^T(\mathbf{k}_b + \bar{\mathbf{M}}\bar{\boldsymbol{\varpi}}^2)\mathbf{q}]_{\mathbf{q}_0}^{\mathbf{q}} - \frac{1}{2}\int_{\mathbf{q}_0}^{\mathbf{q}} \bar{k}(\mathbf{q}^T\mathbf{q})^{-1/2} d(\mathbf{q}^T\mathbf{q}) - \bar{f}_1(q_1 - q_{10}) \\ &= \frac{1}{2}[\bar{m}\bar{\varpi}^2(q_1^2 - q_{10}^2) + \bar{\Omega}^2 q_2^2 + k_b q_2^2] - \bar{k}(\mathbf{q}^T\mathbf{q})^{1/2} + \bar{k}q_{10} - \bar{f}_1 q_1 + \bar{f}_1 q_{10} = \frac{1}{2}[\bar{m}\bar{\varpi}^2(q_1^2 + q_{10}^2) + \bar{\Omega}^2 q_2^2 + k_b q_2^2] - \bar{k}\sqrt{q_1^2 + q_2^2} - \bar{f}_1 q_1.\end{aligned} \tag{2.12}$$

In this equation, the term $k_b q_2^2/2$ represents the elastic energy of the beam, which vanishes if only the beam rigid mode is retained. For the equilibrium point (1), Fig. 3 shows the vertical components of the nonlinear stiffness force and the potential energy of the system as the functions of $\mathbf{q} = [q_1 \quad q_2]^T$ and the added stiffness.

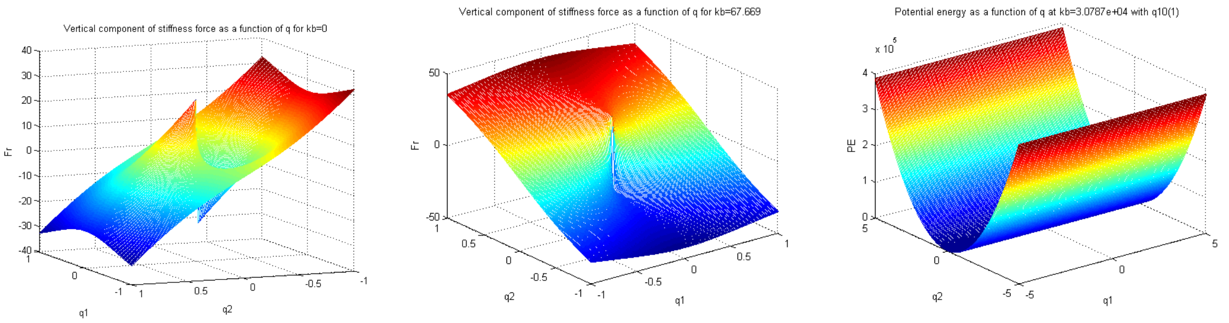

Fig.3 Vertical nonlinear stiffness force and potential energy for equilibrium point (1) affected by the two beam modes.

Kinetic energy: The kinetic energy at position $\mathbf{q}$ of the system is by

$$\mathrm{T}[\mathbf{q}] = \frac{1}{2}\dot{\mathbf{q}}^T(\bar{\mathbf{M}} + \mathbf{m}_b)\dot{\mathbf{q}} = \frac{1}{2}[\bar{m}\dot{q}_1^2 + (1 + m_b)\dot{q}_2^2]. \tag{2.13}$$

The total mechanical energy of undammed system in a solution is a constant, so that we

$$\mathrm{T}[\mathbf{q}] + \Pi[\mathbf{q}] = C. \tag{2.14}$$

Equation governing the influence of nonlinear suspension unit on beam motions

Combining Eqs. 1.12 and 1.13 and eliminating the interaction force vector $\bar{\mathbf{f}}$, we obtain

$$(\bar{\mathbf{m}} + \mathbf{m}_N)\ddot{\bar{\boldsymbol{\Phi}}} + \mathbf{c}_N\dot{\bar{\boldsymbol{\Phi}}} + [\bar{\mathbf{m}}\bar{\boldsymbol{\Lambda}}^2 + \mathbf{k}_N]\bar{\boldsymbol{\Phi}} = \mathbf{c}_{N1}\dot{q}_1 + \mathbf{F}_0, \tag{2.15}$$

$$\mathbf{m}_N = \bar{\mathbf{Y}},\ \ \mathbf{k}_N = \bar{\Omega}^2(1 - \bar{k}/\bar{\mu})\bar{\mathbf{Y}},\ \ \mathbf{c}_N = 2(\mathrm{E}\bar{\Omega} + \bar{\varepsilon}q_2^2/\bar{\mu}^2)\bar{\mathbf{Y}},\ \ \mathbf{c}_{N1} = -2\bar{\varepsilon}q_2 q_1/\bar{\mu}^2\mathbf{Y}_0^T, \tag{2.16}$$

This matrix equation describes the beam dynamics influenced by the nonlinear suspension unit. Here, $\mathbf{m}_N$, $\mathbf{k}_N$, $\mathbf{c}_N$ and $\mathbf{c}_{N1}$ denote an additional mass, stiffness and damping matrix added to the beam by the nonlinear suspension unit, respectively. Due to these added parameters, the dynamic behavior of the beam is as follows.

(i) The linear beam system now behaves nonlinearly due to nonlinear stiffness $\mathbf{k}_N$ and damping $\mathbf{c}_N$ and $\mathbf{c}_{N1}$.

(ii) The couplings between the normalized mode coordinates of beam are generated by the additional mass, stiffness and damping matrices.

(iii) The natural frequencies of the beam about an equilibrium point are affected, which will be discussed later.

Stability and frequencies of small vibrations about equilibrium points

Designs of suspension systems concern two important requirements: 1) the designed system has a stable static equilibrium position in its working environments; 2) the system has a particular required dynamic stiffness, very high or very low, measured by the supporting frequency of small vibrations of the system about the static equilibrium point. To reveal these characteristics, we must investigate the behavior about each equilibrium point. To this end, we can examine the eigenvalues κ_I of the Jacobean matrix of the system at each point from Eq.1.17 or the frequencies $\mathrm{i}\breve{\Omega}_I = \kappa_I$ of small free vibrations of the system about each point using Eq.1.15. Here, we consider vector $\mathbf{Q} = \mathbf{Q}_0 + \tilde{\mathbf{Q}}$ in Eq. 1.15 to derive a linearized equation at point $\mathbf{Q}_0$ which describes small vibrations of the system about this point.

$$\hat{\mathbf{M}}\ddot{\tilde{\mathbf{Q}}} + [\hat{\mathbf{C}}^L + \hat{\mathbf{C}}^N(\mathbf{Q}_0)]\dot{\tilde{\mathbf{Q}}} + [\hat{\mathbf{K}}^L + \hat{\mathbf{K}}^N(\mathbf{Q}_0) + \breve{\mathbf{K}}(\mathbf{Q}_0)]\tilde{\mathbf{Q}} = \hat{\mathbf{F}}_0,\ \ \breve{\mathbf{K}}(\mathbf{q}_0) = \mathbf{T}^T(\frac{\bar{k}}{\bar{\mu}_0^3}\mathbf{q}_0\mathbf{q}_0^T + \frac{\bar{f}_1}{q_{10}}\mathbf{I}_1)\mathbf{T}. \tag{3.1}$$

For an equilibrium point (β), using Eqs.1.14, 1.16 and 3.1, we obtain the following matrices

$$
\begin{aligned}
&\hat{\mathbf{K}}^{L}=\begin{bmatrix}\overline{m}\overline{\omega}^{2} & \mathbf{0}^{T}\\ \mathbf{0} & \overline{\mathbf{m}}\overline{\mathbf{\Lambda}}^{2}+\overline{\Omega}^{2}\overline{\mathbf{Y}}\end{bmatrix},\ \hat{\mathbf{K}}^{N}(\mathbf{q}_{0})=-\frac{\bar{k}}{\overline{\mu}_{0}^{(\beta)}}\begin{bmatrix}1+\bar{f}_{1}/(\eta_{1}^{(\beta)}\bar{k}) & 0\\ 0 & \overline{\mathbf{Y}}\end{bmatrix},\ \hat{\mathbf{C}}^{L}=2\begin{bmatrix}\overline{m}\overline{\omega}E_{1} & \mathbf{0}^{T}\\ \mathbf{0} & \overline{\Omega}\mathrm{E}\overline{\mathbf{Y}}\end{bmatrix},\\
&\breve{\mathbf{K}}(\mathbf{q}_{0})=\frac{\bar{k}}{\overline{\mu}_{0}^{(\beta)}}\begin{bmatrix}\eta_{1}^{(\beta)2}+\bar{f}_{1}/(\eta_{1}^{(\beta)}\bar{k}) & \eta_{1}^{(\beta)}\eta_{2}^{(\beta)}\mathbf{Y}_{0}\\ \eta_{1}^{(\beta)}\eta_{2}^{(\beta)}\mathbf{Y}_{0}^{T} & \eta_{2}^{(\beta)2}\overline{\mathbf{Y}}\end{bmatrix},\ \hat{\mathbf{C}}^{N}(\mathbf{q}_{0})=2\overline{m}\tilde{c}_{1}E_{1}\overline{\omega}\begin{bmatrix}\eta_{1}^{(\beta)2} & \eta_{1}^{(\beta)}\eta_{2}^{(\beta)}\mathbf{Y}_{0}\\ \eta_{1}^{(\beta)}\eta_{2}^{(\beta)}\mathbf{Y}_{0}^{T} & \eta_{2}^{(\beta)2}\overline{\mathbf{Y}}\end{bmatrix},
\end{aligned}
\tag{3.2}
$$

Substituting Eq.3.2 together with the definition of $\overline{m}E_1\overline{\omega}\tilde{c}_1$ and $\tilde{c}_K$ given in Eq. 1.11 into Eq. 3.1, we obtain the following equation describing free vibrations about points (β)

$$
\begin{aligned}
&\hat{\mathbf{M}}\ddot{\hat{\mathbf{Q}}}+2\hat{\mathbf{\Pi}}\dot{\hat{\mathbf{Q}}}+\hat{\mathbf{\Lambda}}\hat{\mathbf{Q}}=\mathbf{0},\quad \hat{\mathbf{\Lambda}}=\begin{bmatrix}\overline{m}\overline{\omega}^{2}-(1-\eta_{1}^{2})\bar{k}/\overline{\mu}_{0} & \eta_{1}\eta_{2}\bar{k}/\overline{\mu}_{0}\,\mathbf{Y}_{0}\\ \eta_{1}\eta_{2}\bar{k}/\overline{\mu}_{0}\,\mathbf{Y}_{0}^{T} & \overline{\mathbf{m}}\overline{\mathbf{\Lambda}}^{2}+[\overline{\Omega}^{2}-(1-\eta_{2}^{2})\bar{k}/\overline{\mu}_{0}]\overline{\mathbf{Y}}\end{bmatrix},\\
&\hat{\mathbf{Q}}=\begin{bmatrix}\tilde{q}_{1} & \tilde{\mathbf{\Phi}}^{T}\end{bmatrix},\quad \hat{\mathbf{M}}=\begin{bmatrix}\overline{m} & \mathbf{0}^{T}\\ \mathbf{0} & \overline{\mathbf{Y}}+\overline{\mathbf{m}}\end{bmatrix},\quad \hat{\mathbf{\Pi}}=\begin{bmatrix}\overline{m}\overline{\omega}E_{1}(1+\tilde{c}_{1}\eta_{1}^{2}) & \overline{m}E_{1}\overline{\omega}\tilde{c}_{1}\eta_{1}\eta_{2}\mathbf{Y}_{0}\\ \overline{\Omega}E\tilde{c}_{K}\eta_{1}\eta_{2}\mathbf{Y}_{0}^{T} & \overline{\Omega}E(1+\tilde{c}_{K}\eta_{2}^{2})\overline{\mathbf{Y}}\end{bmatrix}.
\end{aligned}
\tag{3.3}
$$

where, we have omitted the superscript (β) which is used to identify an equilibrium point listed in Table 1. However, we need to remember that the values of η_1, η_2 and $\overline{\mu}_0$ are related to (β).

Natural vibrations: The natural frequencies of system are governed by the real eigenvalue equations of the system

$$
\left|\hat{\mathbf{\Lambda}}-\hat{\breve{\Omega}}^{2}\hat{\mathbf{M}}\right|=0,\qquad (\hat{\mathbf{\Lambda}}-\hat{\breve{\Omega}}^{2}\hat{\mathbf{M}})\hat{\mathbf{Q}}=0,
\tag{3.4}
$$

from which we obtained the natural frequencies and corresponding modes of the system in the following matrix forms,

$$
\hat{\breve{\mathbf{\Lambda}}}=\mathrm{diag}(\hat{\breve{\Omega}}_{I}),\qquad \mathbf{\Psi}=\begin{bmatrix}\hat{\mathbf{Q}}_{1} & \hat{\mathbf{Q}}_{2} & \cdots & \hat{\mathbf{Q}}_{N+1}\end{bmatrix}\qquad \mathbf{\Psi}^{T}\hat{\mathbf{M}}\mathbf{\Psi}=\mathbf{I},\qquad \mathbf{\Psi}^{T}\hat{\mathbf{\Lambda}}\mathbf{\Psi}=\hat{\breve{\mathbf{\Lambda}}}^{2}.
\tag{3.5}
$$

For ***points*** (1) and (2) with ($\eta_1=\pm1$, $\eta_2=0$), Eq. 3.4 becomes

$$
(\overline{m}\overline{\omega}^{2}-\hat{\breve{\Omega}}^{(1,2)2}\overline{m})\tilde{q}_{1}=0,\qquad \hat{\breve{\Omega}}_{1}^{(1,2)2}=\overline{\omega}^{2},
\tag{3.6}
$$

$$
\{\overline{\mathbf{m}}\overline{\mathbf{\Lambda}}^{2}+[\overline{\Omega}^{2}-\bar{k}/\overline{\mu}_{0}]\overline{\mathbf{Y}}-\hat{\breve{\Omega}}^{(1,2)2}(\overline{\mathbf{Y}}+\overline{\mathbf{m}})\}\tilde{\mathbf{\Phi}}=0.
\tag{3.7}
$$

There is no coupling between the vibrations in the horizontal and vertical directions. Eq. 3.6 directly gives the natural frequency $\hat{\breve{\Omega}}_1^{(1,2)}$ in the horizontal direction of the system. However, Eq. 3.7 still needs to be solved by using numerical methods, which will provide the N natural frequencies $\hat{\breve{\Omega}}_I^{(1,2)}$, $(I=2,3,\cdots,N+1)$, of the beam affected by the nonlinear unit, as defined by Eq. 2.10. Because of no coupling in Eqs. 3.6 and 3.7, the corresponding mode vector matrix takes the following form

$$
\mathbf{\Psi}^{(1,2)}=\begin{bmatrix}\psi_{1} & \mathbf{0}^{T}\\ \mathbf{0} & \tilde{\mathbf{\Psi}}\end{bmatrix}^{(1,2)}.
\tag{3.8}
$$

At ***points*** (2,3) of ($\eta_2\neq0$), the stiffness matrix $\hat{\mathbf{\Lambda}}$ is not diagonal, and there is a coupling between the horizontal and vertical directions, so that we need to solve Eq.3.4 numerically for the natural frequencies $\hat{\breve{\Omega}}_I^{(2,3)}$, $(I=1,2,\cdots,N+1)$, and the corresponding mode matrix $\mathbf{\Psi}^{(2,3)}$ of the system.

Free vibrations: The damping matrix and stiffness matrix in Eq. 3.3 are non-diagonal, except for points (1, 2) with $\eta_2=0$. For convenience of theoretical analysis, we use the mode transformation

$$
\hat{\mathbf{Q}}=\mathbf{\Psi}^{(\beta)}\breve{\mathbf{Q}},\qquad \breve{\mathbf{Q}}=\begin{bmatrix}\breve{q}_{1} & \breve{\mathbf{\Phi}}^{T}\end{bmatrix},\qquad (\beta=1,2,3,4),
\tag{3.9}
$$

to transform Eq. 3.4 into the following form

$$
\ddot{\breve{\mathbf{Q}}}+2\breve{\mathbf{\Pi}}^{(\beta)}\dot{\breve{\mathbf{Q}}}+\hat{\breve{\mathbf{\Lambda}}}^{(\beta)2}\breve{\mathbf{Q}}=\mathbf{0},\qquad \breve{\mathbf{\Pi}}^{(\beta)}=\mathbf{\Psi}^{(\beta)T}\hat{\mathbf{\Pi}}^{(\beta)}\mathbf{\Psi}^{(\beta)},
\tag{3.10}
$$

in which there is only a damping matrix that is non-diagonal. The frequencies of free vibrations and corresponding modes are governed by the following complex eigenvalue problem

$$
\left|\hat{\breve{\mathbf{\Lambda}}}^{(\beta)2}+2\mathrm{i}\breve{\Omega}^{(\beta)}\breve{\mathbf{\Pi}}^{(\beta)}-\breve{\Omega}^{(\beta)2}\mathbf{I}\right|=\mathbf{0},\qquad (\hat{\breve{\mathbf{\Lambda}}}^{(\beta)2}+2\mathrm{i}\breve{\Omega}^{(\beta)}\breve{\mathbf{\Pi}}^{(\beta)}-\breve{\Omega}^{(\beta)2}\mathbf{I})\breve{\mathbf{Q}}=0.
\tag{3.11}
$$

Since the damping matrix $\breve{\mathbf{\Pi}}^{(\beta)}$ is not diagonal, Eq. 3.11 needs to be solved by a numerical method. As an approximation, we may neglect the non-diagonal terms in the damping matrix to obtain the following approximate complex frequencies

$$
\breve{\Omega}_{I}^{(\beta)}\approx\mathrm{i}\breve{E}_{I}^{(\beta)}\hat{\breve{\Omega}}_{I}^{(\beta)}\pm\hat{\breve{\Omega}}_{I}^{(\beta)}\sqrt{1-\breve{E}_{I}^{(\beta)2}},\qquad \breve{E}_{I}^{(\beta)}=\breve{\Pi}_{II}^{(\beta)}/\hat{\breve{\Omega}}_{I}^{(\beta)},\qquad (I=1,2,\cdots,N+1).
\tag{3.12}
$$

For ***points*** (1,2) in association with Eq. 3.9), Eq. 3.10 reduces to

$$
\begin{aligned}
&\breve{\mathbf{\Pi}}^{(1,2)}=\begin{bmatrix}\psi_{1} & \mathbf{0}^{T}\\ \mathbf{0} & \tilde{\mathbf{\Psi}}^{T}\end{bmatrix}^{(1,2)}\begin{bmatrix}\overline{m}\overline{\omega}E_{1}(1+\tilde{c}_{1}) & \mathbf{0}^{T}\\ \mathbf{0} & \overline{\Omega}\mathrm{E}\overline{\mathbf{Y}}\end{bmatrix}\begin{bmatrix}\psi_{1} & \mathbf{0}^{T}\\ \mathbf{0} & \tilde{\mathbf{\Psi}}\end{bmatrix}^{(1,2)}=\begin{bmatrix}\breve{\Pi}_{11}^{(1,2)} & \mathbf{0}^{T}\\ \mathbf{0} & \tilde{\mathbf{\Pi}}^{(1,2)}\end{bmatrix},\\
&\breve{\Pi}_{11}^{(1,2)}=\overline{m}\overline{\omega}E_{1}(1+\tilde{c}_{1})\psi_{1}^{(1,2)2},\qquad \tilde{\mathbf{\Pi}}^{(1,2)}=\overline{\Omega}\mathrm{E}\tilde{\mathbf{\Psi}}^{(1,2)T}\overline{\mathbf{Y}}\tilde{\mathbf{\Psi}}^{(1,2)}.
\end{aligned}
\tag{3.13}
$$

As a result of this, Eq.3.11 becomes

$$(\hat{\breve{\Omega}}^{(1,2)2} + 2\mathrm{i}\breve{\Omega}^{(1,2)}\breve{\Pi}_{11}^{(1,2)} - \breve{\Omega}^{(1,2)2})\breve{q}_1 = 0, \tag{3.14}$$

$$\{\hat{\breve{\mathbf{\Lambda}}}_\phi^2 + 2\mathrm{i}\breve{\Omega}^{(1,2)}\tilde{\mathbf{\Pi}}^{(1,2)} - \breve{\Omega}^{(1,2)2}\mathbf{I}\}\breve{\mathbf{\Phi}} = \mathbf{0}, \qquad \hat{\breve{\mathbf{\Lambda}}}_\phi^{(1,2)} = \mathrm{diag}(\hat{\breve{\Omega}}_{I=2,3,\cdots,N+1}). \tag{3.15}$$

From these equations, we obtain the corresponding complex frequencies of free vibrations about points (1, 2) as

$$\breve{\Omega}_1^{(1,2)} = \mathrm{i}\breve{E}_1^{(1,2)}\hat{\breve{\Omega}}_1^{(1,2)} \pm \hat{\breve{\Omega}}_1^{(1,2)}\sqrt{1-\breve{E}_1^{(1,2)2}}, \qquad \breve{E}_1^{(1,2)} = \breve{\Pi}_{11}^{(1,2)} / \hat{\breve{\Omega}}^{(1,2)}. \tag{3.16}$$

$$\breve{\Omega}_I^{(1,2)} \approx \mathrm{i}\breve{E}_I^{(1,2)}\hat{\breve{\Omega}}_I^{(1,2)} \pm \hat{\breve{\Omega}}_I^{(1,2)}\sqrt{1-\breve{E}_I^{(1,2)2}}, \qquad \breve{E}_I^{(1,2)} = \tilde{\Pi}_{II}^{(1,2)} / \hat{\breve{\Omega}}_I^{(1,2)}, \qquad (I = 2,\cdots,N+1). \tag{3.17}$$

The solution for retaining only rigid mode: For an analytical analysis, we retain only the rigid mode of beam, so that

$$\mathbf{Y}_0 = 1, \ \ \overline{\mathbf{Y}} = 1, \ \ \overline{\mathbf{m}} = \tilde{\rho}\tilde{S}, \ \ \overline{\mathbf{\Lambda}}^2 = \hat{\overline{\Omega}}_1^2 = 0, \ \ \tilde{\mathbf{\Phi}} = \tilde{q}_2, \tag{3.18}$$

and thus Eq.3.4 reduces to

$$\begin{bmatrix} \overline{m} & 0 \\ 0 & \overline{M} \end{bmatrix}\begin{bmatrix} \ddot{\tilde{q}}_1 \\ \ddot{\tilde{q}}_2 \end{bmatrix} + 2\begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix}\begin{bmatrix} \dot{\tilde{q}}_1 \\ \dot{\tilde{q}}_2 \end{bmatrix} + \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix}\begin{bmatrix} \tilde{q}_1 \\ \tilde{q}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \overline{M} = 1 + \tilde{\rho}\tilde{S},$$
$$\Pi_{11} = \overline{m}\overline{\omega}E_1(1+\tilde{c}_1\eta_1^2), \ \ \overline{m}E_1\overline{\omega}\tilde{c}_1\eta_1\eta_2 = \Pi_{12} = \Pi_{21} = \overline{\Omega}\mathrm{E}\tilde{c}_K\eta_1\eta_2, \ \ \Pi_{22} = \overline{\Omega}\mathrm{E}(1+\tilde{c}_K\eta_2^2),$$
$$\Lambda_{11} = \overline{m}\overline{\omega}^2 - (1-\eta_1^2)\overline{k}/\overline{\mu}_0, \ \ \Lambda_{12} = \Lambda_{21} = \eta_1\eta_2\overline{k}/\overline{\mu}_0, \ \ \Lambda_{22} = \overline{\Omega}^2 - (1-\eta_2^2)\overline{k}/\overline{\mu}_0. \tag{3.19}$$

The two natural frequencies obtained by solving the corresponding Eq. 3.5 are as follows

$$\hat{\overline{\Omega}}^{(\beta)2} = \{(\overline{\Lambda}_{11}+\overline{\Lambda}_{22}) \pm \sqrt{(\overline{\Lambda}_{11}+\overline{\Lambda}_{22})^2 - 4(\overline{\Lambda}_{11}\overline{\Lambda}_{22}-\overline{\Lambda}_{12}^2)}\}/2, \qquad \overline{\Lambda}_{11} = \Lambda_{11}/\overline{m}, \qquad \overline{\Lambda}_{22} = \Lambda_{22}/\overline{M}, \qquad \overline{\Lambda}_{12} = \Lambda_{12}/\sqrt{\overline{M}\overline{m}}. \tag{3.20}$$

Points $(\beta = 1,2)$: $\Lambda_{12} = 0 = \overline{\Lambda}_{12}$ due to $\eta_2 = 0$, we obtain

$$\hat{\overline{\Omega}}_1^{(1,2)2} = \overline{\Lambda}_{11} = \Lambda_{11}/\overline{m}, \qquad \hat{\overline{\Omega}}_1^{(1,2)} = \sqrt{\Lambda_{11}/\overline{m}}; \qquad \hat{\overline{\Omega}}_2^{(1,2)2} = \overline{\Lambda}_{22} = \Lambda_{22}/\overline{M}, \qquad \hat{\overline{\Omega}}_2^{(1,2)} = \sqrt{\Lambda_{22}/\overline{M}}, \tag{3.21}$$

which represents the natural frequencies of small vibrations of the system about points (1,2) in the horizontal and vertical directions, respectively.

Points $(\beta = 2,3)$: since $\overline{\Lambda}_{12} > 0$, we have

$$\sqrt{(\overline{\Lambda}_{11}+\overline{\Lambda}_{22})^2 - 4(\overline{\Lambda}_{11}\overline{\Lambda}_{22}-\overline{\Lambda}_{12}^2)} > \sqrt{(\overline{\Lambda}_{11}+\overline{\Lambda}_{22})^2 - 4\overline{\Lambda}_{11}\overline{\Lambda}_{22}} = \left|\overline{\Lambda}_{11}-\overline{\Lambda}_{22}\right|, \tag{3.22}$$

so that the solutions $\hat{\overline{\Omega}}^{(\beta)2}$ of Eq. 3.20 satisfy the following equation

$$\hat{\overline{\Omega}}_1^{(2,3)2} < 0.5\{(\overline{\Lambda}_{11}+\overline{\Lambda}_{22}) - \sqrt{(\overline{\Lambda}_{11}-\overline{\Lambda}_{22})^2}\} = \mathrm{Min}(\hat{\overline{\Omega}}_1^{(1,2)2}, \hat{\overline{\Omega}}_2^{(1,2)2}),$$
$$\hat{\overline{\Omega}}_2^{(2,3)2} > 0.5\{(\overline{\Lambda}_{11}+\overline{\Lambda}_{22}) + \sqrt{(\overline{\Lambda}_{11}-\overline{\Lambda}_{22})^2}\} = \mathrm{Max}(\hat{\overline{\Omega}}_1^{(1,2)2}, \hat{\overline{\Omega}}_2^{(1,2)2}). \tag{3.23}$$

This implies that the lower natural frequency about points $(\beta = 2,3)$ is smaller than the lower natural frequency about points (1,2) but the higher one about points $(\beta = 2,3)$ is larger than the higher one about point (1).

Stability: The eigenvalues κ_I of the Jacobean matrix of Eq.1.17 at a point (β) are given by

$$\hat{\breve{\kappa}}_I^{(\beta)} = \mathrm{i}\hat{\breve{\Omega}}_I^{(\beta)}, \qquad \breve{\kappa}_I^{(\beta)} = \mathrm{i}\breve{\Omega}_I^{(\beta)}, \tag{3.24}$$

for natural vibrations and free vibrations, respectively. From Eq.3.12, we obtain the approximate eigenvalues

$$\breve{\kappa}_I^{(\beta)} \approx -\breve{E}_I^{(\beta)}\hat{\breve{\Omega}}_I^{(\beta)} \pm \mathrm{i}\hat{\breve{\Omega}}_I^{(\beta)}\sqrt{1-\breve{E}_I^{(\beta)2}}, \qquad \breve{E}_I^{(\beta)} = \breve{\Pi}_{II}^{(\beta)} / \hat{\breve{\Omega}}_I^{(\beta)}, \qquad (I = 1,2,\cdots,N+1). \tag{3.25}$$

The eigenvalues in Eq.3.25 have only negative real parts which confirm that the system is stable at point (β). The characteristic of eigenvalues in Eq.3.25 depends on the natural frequency $\hat{\breve{\Omega}}_I^{(\beta)}$ and damping $\breve{E}_I^{(\beta)}$ which are determined by Eq.3.4. For majority of engineering systems, there are a real natural frequency $\hat{\breve{\Omega}}_I^{(\beta)}$ and small positive damping $\breve{E}_I^{(\beta)} < 1$, so that the free vibrations of the system about point (β) are stable damped vibrations.

Engineering Applications

Nonlinear suspension system with extreme low supporting frequency for GVT of aircrafts

We consider Fig. 1 as a model for GVT of large full-scale aircrafts. The central mass is considered as the fuselage and the two beams are the two wings of aircraft. Aircrafts flying in the air are in a free-free state without any mechanical supports. However, in tests, the aircraft is supported on the ground so that the supporting system must affect its dynamic characteristics. It has demonstrated that the effect of supporter on the aircraft could be neglected if the frequency Ω_{SA} of an assumed rigid aircraft on the supporter is less than one third of the first elastic natural frequency Ω_{EA} of the free-free aircraft [1-2], i.e.

$$\Omega_{SA} \le \Omega_{EA}/3 \tag{4.1}$$

For small aircrafts, their first elastic natural frequency is high enough and there is no difficulty to design a supporting system satisfying Eq. 4.1. However, for very large aircrafts, the first natural frequency is lower than 1Hz. Therefore, the supporting frequency for GVT should be less than 0.3 Hz. Due to a very large weight of the aircraft, the static stiffness of supporter have to be sufficiently large. Therefore, it is very difficult to design a supporting system with less than 0.3Hz supporting frequency for large aircrafts by means of normal supporting designs[1-2]. Non-linear supporting system discussed in this paper provides an effective approach to design this type of supporting systems.

To support the airplane on the ground, Point (1) or (2) is chosen as the static equilibrium state of aircraft on the ground. Based on Eq.1.1, $\Delta = Y_0 - L,$ from which we can choose a suitable height Y_0 according to the initial length L of the vertical spring, so that a negative parameter $\bar{\Delta}$ and the vertical stiffness $\bar{K}$ are determined by

$$\bar{K} = (1+\bar{\rho}\bar{S})\bar{g}/\left|\bar{\Delta}\right|, \qquad K = M\Omega_0^2\bar{K}. \tag{4.2}$$

Because the total mass of the large airplane is huge, the stiffness of the vertical spring is very large. If there are no two inclined springs, the supporting frequency of the aircraft and the static compression of the vertical spring are required to satisfy the conditions

$$\Omega_{SA} = \sqrt{\bar{K}/(1+\bar{\rho}\bar{S})} = \sqrt{\bar{g}/\left|\bar{\Delta}\right|} \le \Omega_{EA}/3, \qquad \left|\bar{\Delta}\right| \ge 9\bar{g}/\Omega_{Ea}^2. \tag{4.3}$$

A limited space of test site does not allow these conditions to be realized only using linear supporting systems. From Eq.3.21 and Table 1, we have the natural frequency

$$\Omega_{NSA} = \hat{\bar{\Omega}}_2^{(1,2)} = \sqrt{(1-\bar{k}\bar{\Omega}^{-2}/\bar{\mu}_0^{(1,2)})/(1+\tilde{\rho}\tilde{S})}\bar{\Omega}, \quad \bar{\mu}_0^{(1,2)} = \tilde{k}_1 + \tilde{K}_1\bar{\Delta}_1. \tag{4.4}$$

Based on Eqs.1.7 and 1.11, we obtain

$$\bar{\Delta}_1 = (X_0 - L_1)/l, \qquad \bar{\mu}_0^{(1,2)} = \tilde{K}_1(k/K_1 + \bar{\Delta}_1), \tag{4.5}$$

so that to reduce the supporting frequency in Eq. 4.4, we have to choose point (1) to design the system satisfying

$$1 \ge \bar{k}\bar{\Omega}^{-2}/\bar{\mu}_0^{(1,2)} > 0, \qquad \tilde{k}_1 + \tilde{K}_1\bar{\Delta}_1 > 0. \tag{4.6}$$

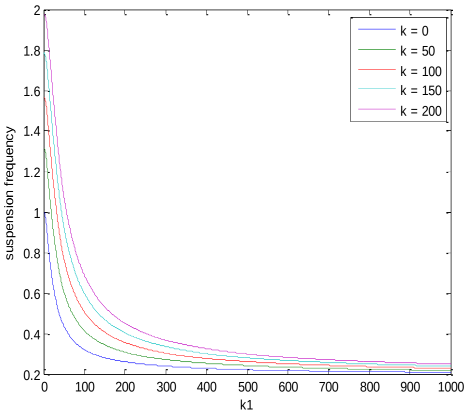

Fig.4. Suspension frequency affected by supporting stiffness

This design condition is easily realized. In this design, the nonlinear stiffness term $\bar{k}\bar{\Omega}^{-2}/\bar{\mu}_0^{(1)}$ plays a negative stiffness in Eq.4.4 to reduce the supporting frequency. Theoretically, from Eq.4.6, we may choose the value of $\bar{k}\bar{\Omega}^{-2}/\bar{\mu}_0^{(1)}$ near to 1 to realise the standard $\Omega_{NSA} < \Omega_{EA}/3$. Eq. 3.19 confirms that for the system with two dampers in Fig. 1, the damping factors are positive and the system is stable. Therefore, during GVT involving small vibrations of the aircraft about point (1), any small disturbance of the system from the equilibrium state can be damped. Fig. 4 shows the supporting frequency affected by stiffness values of two springs.

A rigid supporting platform

Now we consider the system shown in Fig. 1 as a model of laboratory dynamic tests. We wish that the tested beam is fixed on the "rigid" foundation with an extreme large supporting stiffness. We also choose stable point (1) or (2) as our static equilibrium state of the system to obtain an extreme high supporting frequency given in Eq. 4.4. From Eq. 4.5, it is necessary to choose point (2) satisfying

$$\bar{k}\bar{\Omega}^{-2}/\bar{\mu}_0^{(1,2)} < 0, \qquad \tilde{k}_1 + \tilde{K}_1\bar{\Delta}_1 < 0. \tag{4.7}$$

In this design, the nonlinear stiffness term $\bar{k}\bar{\Omega}^{-2}/\bar{\mu}_0^{(2)}$ plays a positive stiffness in Eq.4.4 to increase the supporting frequency. Theoretically, choosing a very small negative value of $\tilde{k}_1 + \tilde{K}_1\bar{\Delta}_1$ satisfying Eq.4.7, we can obtain a sufficient large supporting frequency to treat the supporting foundation as "rigid".

Conclusions

This paper has investigated a generalised integrated structure-nonlinear isolator interaction system consisting of a linear beamlike structure and a geometrical nonlinear isolator, which is able to provide extreme high or low supporting frequency that is not possible using passive linear isolators. The generalised system can be reduced to several simplified systems reported in the available references. The governing equations describing the interaction dynamics are derived, based on which, equilibria with stabilities and small vibration characteristics about each equilibrium point are discussed for engineering designs. The coupling mechanisms of the system are revealed. Two designs, one for GVT of large aircrafts requiring very low supporting frequency and another for vibration tests in laboratories expecting a rigid foundation, are presented. The proposed coupling nonlinear system and the developed theory have established a fundamental basis for vibration isolation designs, and further analytical and numerical investigations on its more complex nonlinear behavior such as bifurcation and chaos which have not discussed in this paper, which is aimed mainly to coupling analysis and particular suspension designs.

References

[1] Molyneux, W. G. (1958) The support of an aircraft for ground resonance tests. *A. E.*, June, 160-172.

[2] Xing, J. T. (1975) Theory and techniques on mode vibration experiments of aircraft structures, Lecture Notes, (in Chinese). Nanjing: NAI Press.

[3] Harris, C. M. & Crede, C. E. (1961) Shock and vibration handbook. McGraw-Hill.

[4] Rivin, E. I. (2003) Passive vibration isolation. New York: ASME Press.

[5] Winterwood, J. (2001) High performance vibration isolation for gravitational wave detection, PhD thesis, University of Western Australia.

[6] Fuller, C. R., Elliott, S. J. & Nelson, P. A. (1996) Active control of vibration. Academic Press.

[7] Xing, J. T., Xiong, Y. P. & Price, W. G. (2005) Passive-active vibration isolation systems to produce zero or infinite dynamic modulus: theoretical and conceptual design strategies. *J. S. & Vib.*, 286, 615-636.

[8] Alabuzhev, P., Gritchin, Kim, A. L., Migirenko, G. Chon, V. & Stepanov, P. (1989) Vibration protecting and measuring systems with quasi-zero stiffness. New York: Hemisphere.

[9] Platus, D. L. (1992) Negative-stiffness-mechanism vibration isolation system. In *Proc. of SPIE-The Int. Soc. for Optical Eng., Vib. Control in Microelectronics, Optics, and Metrology,* Vol. 1619, pp.44-54. San Jose, CA, USA.

[10] Zhang, J. Z., Li, D., Chen, M. J. & Dong, S. (2004) An ultra-low frequency parallel connection nonlinear isolation for precision instruments. *Key Engineering Materials*, 257-258, 231-236.

[11] Carrela, A., Brennan, M. J. & Waters, T. P. (2007) Static analysis of a passive vibration isolator with quasi-zero-stiffness characteristic. *J. S. & Vib.*, 301, 678-689.

[12] Carrela, A., Brennan, M. J., Waters, T. P. & Shin, K. (2008) On the design of a high-static-low-dynamic stiffness isolator using linear mechanical springs and magnets. *J. S. & Vib.*, 315, 712-720.

[13] Kovaic, I., Brennan, M.,J. & Waters, T. P. (2008) A study of a nonlinear vibration isolator with a quasi-zero stiffness characteristic. *J. S. & Vib.*, 315, 700-711.

[14] Ahn, H.-J. (2008) Performance limit of a passive vertical isolator using a negative stiffness mechanism, *Journal of Mechanical Science & Technology*. 22, 2357-2364.

[15] Cao, Q., Wiercigroch, M., Pavlovskaia, E. E., Grebogi C. & Thompson, J.M.T. (2008) The limit case response of the archetypal oscillator for smooth and discontinuous dynamics. *Int. J. Non. Mech.*, 43, 462-473.

[16] Cao, Q., Wiercigroch, M., Pavlovskaia, E.E, Grebogi, C. & Thompson, J.M.T. (2008) Piecewise linear approach to an archetypal oscillator for smooth and discontinuous dynamics, *Phil. Trans. Roy. Soc. A***366**, 635-652.

[17] Liu, Y., Chen, Y. & Cao, Q. (2010) The bifurcations of resonance in irrational system. *Shock & Vibration*.

[18] Xing, J.T., Xiong, Y.P. & Price, W.G. (2009) A generalised mathematical model and analysis for integrated multi-channel vibration structure-control interaction systems. *J. S. & Vib.*, 320, 584-616.

[19] Abraham, R. & Marsden, J. E. (1978) Foundations of mechanics. Massachusetts: The Benjamin / Cummings Publishing Company, INC.

[20] Guckenheimer, J. & Holmes, P. (1983) Nonlinear oscillations, dynamical systems, and bifurcations of vector fields. Springer-Verlag.

[21] Thompson, J.M.T. & Stewart, H.B. (1986) Nonlinear dynamics and chaos, geometrical methods for engineers and scientists. Chichester: John Wiley & Sons.

[22] Thomson, W. T. (1988) Theory of vibration with applications. 3rd edn. Prentice-Hall.

[23] Xing, J.T. & Price, W.G. (2004) A substructure method for power flow analysis of non-linear systems consisting of linear substructures and non-linear controllers. In *Proceedings of the Eleventh International Congress on Sound and Vibration* (CD-ROM), pp.2657-2664. St. Petersburg, Russian, 5-8 July 2004, Published by IIAV.

[24] Xing, J.T. (1986) A study on finite element method and substructure-subdomain technique for dynamic analysis of coupled fluid-solid interaction problems, (in Chinese). *Acta Mechanica Solida Sinica*, 12, 329-337

[25] Xing, J.T. & Price, W.G. (1991) A mixed finite element method for the dynamic analysis of coupled fluid-solid interaction problems. *Proc. R. Soc. Lond. A***433**, 235-255.

[26] Xing, J.T., Price, W.G. & Du, Q.H. (1996) Mixed finite element substructure-subdomain methods for the dynamic analysis of coupled fluid-solid interaction problems. *Phil. Trans. R. Soc. Lond. A***354**, 259-295.

[27] Xing, J.T., Price, W.G., Pomfret, M.J. & Yam. L.H. (1997) Natural vibration of a beam-water interaction system. *J. S. & Vib.*, 199, 491-512.

[28] Jin, J.Z. & Xing, J.T. (2007) The transient dynamic analysis of a floating beam - water interaction system excited by the impact of a landing beam. *J. S. & Vib.*, 303, 371-390.

[29] Xing, J.T., Price, W.G. & Chen, Y.G. (2003) A mixed finite-element finite-difference method for nonlinear fluid-structure interaction dynamics. I. Fluid-rigid structure interaction. *Proc. R. Soc. Lond. A***459**, 2399-2430.

[30] Xing, J.T., Price, W.G. & Chen, Y.G. (2005) A numerical method to simulate nonlinear fluid-rigid structure interaction problems. *Acta Mechanica Solida Sinica*, 18, 95-109.

[31] Xiong, Y.P., Xing, J.T. & Price, W.G. (2005) Interactive power flow characteristics of an integrated equipment—nonlinear isolator—travelling flexible ship excited by sea waves. *J. S. & Vib.*, 287, 245-276.

[32] Xing, J.T., Xiong, Y.P., Wiercigroch, M. & Cao, Q. (2011) Mathematical modelling of an integrated converter for wave energy harvesting. In Proceedings of 7th European Nonlinear Dynamics Conference (ENOC2011), (ed D. Bernardini, G. Rega & F. Romeo), Section MS14: Paper 5, P1-6.

[33] Yang, J., Xiong, Y.P. & Xing, J.T. (2013) Dynamics and power flow behavior of a nonlinear vibration isolation system with a negative stiffness mechanism. *J. S. & Vib*. 332, 167-183.

[34] Xing, J.T. (2015) Energy flow theory of nonlinear dynamical systems with applications. Springer.