On the Effect of the Deformed State of a Tire on the Combined Wheel's Rolling, Sliding, and Spinning with Dry Friction

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Summary. The shimmy theories based on the Keldysh assumption can be easily implemented analytically and are still quite efficient for the preliminary analysis of the stability of steady state rolling regimes with no slip and spin of tires. On the other hand, such a shimmy theory uses the non-holonomic rolling model; therefore it is inconsistent with unsteady rolling regimes characterized by the non-vanishing sliding and spin. The qualitatively different model accounting the dry friction effect on the stability of motion is constructed on the groundwork of the coupled dry friction theory. This model has shown its' applicability to some practical problems of engineering design even if a wheel is assumed to be rigid. Here the improved model accounting for the tire deforming, the complex contact pressure distribution and the anisotropy of the dry friction coefficient in case of the combined kinematics is presented.

The Local Model of the Anisotropic Dry Friction -Differential Formulation

In general, the plane-parallel relative motion, i. e. the simultaneous sliding and spinning, of the rigid bodies with the finite contact spot S requires the qualitative improvement of the Amonton-Coulomb dry friction law. The aim of this theory consists in the differential formulation of the Coulomb law as a local model of the friction interaction in each point of the contact area S: $\forall M \in S, \boldsymbol{\tau} = -|\sigma_\nu| \frac{\mathbf{f}\cdot\mathbf{v}_\Sigma}{|\mathbf{v}_\Sigma|}(\mathbf{v}_\Sigma \neq 0)$, $\mathbf{v}_\Sigma = \mathbf{v}_0 - R\boldsymbol{\omega}_\tau \times \mathbf{e}_3 + \boldsymbol{\omega}_\nu \times \mathbf{r}_\tau$. Here $\mathbf{v}_\Sigma$ denotes the summary velocity of the relative slip in the arbitrary point $M \in S$, $\mathbf{v}_0(\mathbf{q})$ is the longitudinal absolute velocity, $\boldsymbol{\omega}_\tau(\mathbf{q})$ is the angular velocity of rolling, , $\boldsymbol{\omega}_\nu(\mathbf{q})$ is the angular velocity of spinning, $R(M)$ is the curvature radius of the rolling body calculated in the point M, $\mathbf{r}_\tau(M)$ is the vector radius of the point $M \in S$ in the plane of contact, $\mathbf{e}_3$ denotes the normal unit vector of the contact plane, $\boldsymbol{\tau}$ is the frictional tangential stress in the contact area S, and σ_ν denotes the normal contact pressure. Thus, the cohesion condition can be formulated locally in the point $M \in S$ as follows: $I_2^{-2}(\mathbf{f})\left\{I_1^2(\mathbf{f})|\boldsymbol{\tau}|^2 + \left[I_1(\mathbf{f})\mathbf{f}_S + \mathbf{f}^{\mathrm{T}}\cdot\mathbf{f}\right] : \boldsymbol{\tau}\otimes\boldsymbol{\tau}\right\} = |\sigma_\nu|^2$.

Considering the combined kinematics, taking into account the dry friction anisotropy, we obtain the following formula for the tangential stress: $\boldsymbol{\tau}_1 = -|\sigma_\nu||\mathbf{v}_0 + \boldsymbol{\omega}_\nu \times \mathbf{r}_\tau|^{-1}\mathbf{f}\cdot(\mathbf{v}_0 - R\boldsymbol{\omega}_\tau \times \mathbf{e}_3 + \boldsymbol{\omega}_\nu \times \mathbf{r}_\tau)$, where the normal pressure accounting for the rolling effect is represented by the linear approximation: $\sigma_\nu = \sigma_0\left[1 + \left(\mathbf{r}_\tau \times \mathbf{h}\cdot\boldsymbol{\omega}_\tau / |\boldsymbol{\omega}_\tau|\right)\cdot\mathbf{e}_3\right]$. Here $\sigma_0 = \sigma_\nu(\boldsymbol{\omega}_\tau = 0)$, and $\mathbf{h} = h_{\alpha\beta}\mathbf{e}^\alpha\mathbf{e}^\beta$ is the "rolling friction tensor" for the anisotropic elastic body; we assume it being homogeneous and positively defined: $\mathbf{h} \neq \mathbf{h}(M)$; $\quad \forall\boldsymbol{\omega}_\tau = \boldsymbol{\omega}_\tau(\mathbf{q}) \quad \boldsymbol{\omega}_\tau^{\mathrm{T}}\cdot\mathbf{h}\cdot\boldsymbol{\omega}_\tau > 0$.

Accounting the rolling effect on the contact pressure that accounts by-turn both sliding (this term is denoted as $\boldsymbol{\tau}_1$) and spinning (this one is denoted as $\boldsymbol{\tau}_2$), we obtain finally the local model of the anisotropic dry friction in case of the combined kinematics $\boldsymbol{\tau} = \boldsymbol{\tau}_1 + \boldsymbol{\tau}_2$:

$$\boldsymbol{\tau}_1 = -|\sigma_0|\left[1 + \left(\mathbf{r}_\tau \times \frac{\mathbf{h}\cdot\boldsymbol{\omega}_\tau}{|\boldsymbol{\omega}_\tau|}\right)\cdot\mathbf{e}_3\right]\frac{\mathbf{f}\cdot(\mathbf{v}_0 - R\boldsymbol{\omega}_\tau \times \mathbf{e}_3)}{|\mathbf{v}_0 - R\boldsymbol{\omega}_\tau \times \mathbf{e}_3 + \boldsymbol{\omega}_\nu \times \mathbf{r}_\tau|};\ \boldsymbol{\tau}_2 = -|\sigma_0|\left[1 + \left(\mathbf{r}_\tau \times \frac{\mathbf{h}\cdot\boldsymbol{\omega}_\tau}{|\boldsymbol{\omega}_\tau|}\right)\cdot\mathbf{e}_3\right]\frac{\mathbf{f}\cdot(\boldsymbol{\omega}_\tau \times \mathbf{r}_\tau)}{|\mathbf{v}_0 - R\boldsymbol{\omega}_\tau \times \mathbf{e}_3 + \boldsymbol{\omega}_\nu \times \mathbf{r}_\tau|}.$$

The global model of the anisotropic dry friction under combined kinematics

The dynamic interaction of the slightly deformed rigid body with the rough support plane is defined by the normal reaction $\mathbf{N}$, the resultant vector of tangent forces $\mathbf{T}$, the anti-rolling couple $\mathbf{M}_\tau$, and the dry friction torque $\mathbf{M}_\nu$. These quantities are obtained by integration of the normal contact stress as well as the summary tangential stress over the contact area S: $\mathbf{N} = \int_S \sigma_\nu\mathbf{e}_3 dS = \int_S \sigma_0\left[\mathbf{e}_3 + \frac{\mathbf{r}_\tau \times (\mathbf{h}\cdot\boldsymbol{\omega}_\tau)}{|\boldsymbol{\omega}_\tau|}\right]dS$; $\mathbf{M}_\tau = \int_S \sigma_\nu\mathbf{r}_\tau \times \mathbf{e}_3 dS = \int_S \sigma_0\mathbf{r}_\tau \times \left[\mathbf{e}_3 + \frac{\mathbf{r}_\tau \times (\mathbf{h}\cdot\boldsymbol{\omega}_\tau)}{|\boldsymbol{\omega}_\tau|}\right]dS$,

$$\mathbf{T} = -\int_S \boldsymbol{\tau}dS = -\int_S \sigma_0\left[1 + \frac{\mathbf{r}_\tau \times (\mathbf{h}\cdot\boldsymbol{\omega}_\tau)}{|\boldsymbol{\omega}_\tau|}\cdot\mathbf{e}_3\right]\frac{\mathbf{f}\cdot(\mathbf{v}_0 - R\boldsymbol{\omega}_\tau \times \mathbf{e}_3 + \boldsymbol{\omega}_\nu \times \mathbf{r}_\tau)}{|\mathbf{v}_0 + \boldsymbol{\omega}_\nu \times \mathbf{r}_\tau|}dS;$$

$$\mathbf{M}_\nu = -\int_S \mathbf{r}_\tau \times \boldsymbol{\tau} dS = -\int_S \sigma_0 \left[1 + \frac{\mathbf{r}_\tau \times (\mathbf{h}\cdot\boldsymbol{\omega}_\tau)}{|\boldsymbol{\omega}_\tau|}\cdot \mathbf{e}_3\right] \frac{\mathbf{r}_\tau \times \left[\mathbf{f}\cdot(\mathbf{v}_0 - R\boldsymbol{\omega}_\tau \times \mathbf{e}_3 + \boldsymbol{\omega}_\nu \times \mathbf{r}_\tau)\right]}{|\mathbf{v}_0 + \boldsymbol{\omega}_\nu \times \mathbf{r}_\tau|} dS.$$

In these formulas, the resultant force $\mathbf{N}_0$ of the static contact pressure σ_0 and it's variation $\mathbf{N}_1$ induced by the rolling effect can be written as follows: $\mathbf{N}_0 = N_0\mathbf{e}_3, \quad N_0 = \int_S \sigma_0 dS;\;\; \mathbf{N}_1 = \int_S \sigma_0 \frac{\mathbf{r}_\tau \times (\mathbf{h}\cdot\boldsymbol{\omega}_\tau)}{|\boldsymbol{\omega}_\tau|} dS = N_1\mathbf{e}_3, \quad N_1 = -\mathbf{S}_\sigma \cdot \mathbf{h}\cdot \frac{\boldsymbol{\omega}_\tau}{|\boldsymbol{\omega}_\tau|}$.

We have the similar formulae for the anti-rolling couple where the "static" anti-rolling couple (that vanishes not in case of the rolling asymmetry of the body) and the "dynamic" one are defined by the formulae $\mathbf{M}_\tau^0 = \int_S \sigma_0 \mathbf{r}_\tau \times \mathbf{e}_3 dS = \mathbf{S}_\sigma$ and $\mathbf{M}_\tau^1 = \int_S \sigma_0 \mathbf{r}_\tau \times \frac{\mathbf{r}_\tau \times (\mathbf{h}\cdot\boldsymbol{\omega}_\tau)}{|\boldsymbol{\omega}_\tau|} dS = -\mathbf{J}_\sigma \cdot \mathbf{h}\cdot \frac{\boldsymbol{\omega}_\tau}{|\boldsymbol{\omega}_\tau|}$ respectively.

Here the first moment vector $\mathbf{S}_\sigma$ and the inertia moment tensor $\mathbf{J}_\sigma$ of the plane area of contact S with the static contact pressure distribution σ_0 are introduced: $\mathbf{S}_\sigma = \int_S \sigma_0 \mathbf{r}_\tau \times \mathbf{e}_3 dS = \int_S \sigma_0 \epsilon_{\alpha\beta}\xi^\beta \mathbf{e}^\alpha dS$, $\mathbf{J}_\sigma = \int_S (\wedge \mathbf{r}_\tau)\otimes(\mathbf{r}_\tau \wedge) dS = \int_S \epsilon_{\alpha\beta}\xi^\beta \epsilon_{\gamma\delta}\xi^\delta \mathbf{e}^\alpha \otimes \mathbf{e}^\beta dS$, $\epsilon_{\alpha\beta} \equiv \epsilon_{\alpha\beta 3} = (\mathbf{r}_\alpha \times \mathbf{r}_\beta)\cdot \mathbf{e}_3, \quad \alpha,\beta,\gamma,\delta = 1,2.$

The homogeneity of the tensors lead to the vanishing as well normal reaction $\mathbf{N}_1$ as the rolling initiation moment $\mathbf{M}_\tau^0$ in the frame $O\xi^1\xi^2$ attached to the center of the figure S, therefore we have the following formulae for the normal reaction and anti-rolling couple: $\mathbf{N} = N_0\mathbf{e}_3; \quad \mathbf{M}_\tau = -\mathbf{J}_\sigma \cdot \mathbf{h}\cdot(\boldsymbol{\omega}_\tau / |\boldsymbol{\omega}_\tau|)$.

The resultant vector of the anisotropic dry friction under combined kinematics can be now expressed through the sum of the following terms: $\mathbf{T}_1 = -\int_S \sigma_0 \frac{\mathbf{f}\cdot\mathbf{v}_S}{|\mathbf{v}_S + \boldsymbol{\omega}_\nu \times \mathbf{r}_\tau|} dS$, $\mathbf{T}_2 = -\int_S \sigma_0 \left[\frac{\mathbf{r}_\tau \times (\mathbf{h}\cdot\boldsymbol{\omega}_\tau)}{|\boldsymbol{\omega}_\tau|}\cdot \mathbf{e}_3\right] \frac{\mathbf{f}\cdot\mathbf{v}_S}{|\mathbf{v}_S + \boldsymbol{\omega}_\nu \times \mathbf{r}_\tau|} dS$, $\mathbf{T}_3 = -\int_S \sigma_0 \frac{\mathbf{f}\cdot(\boldsymbol{\omega}_\nu \times \mathbf{r}_\tau)}{|\mathbf{v}_S + \boldsymbol{\omega}_\nu \times \mathbf{r}_\tau|} dS$. The torque of the anisotropic dry friction under combined kinematics is also represented as a sum of four terms: $\mathbf{M}_1 = -\int_S \sigma_0 \frac{\mathbf{r}_\tau \times (\mathbf{f}\cdot\mathbf{v}_S)}{|\mathbf{v}_S + \boldsymbol{\omega}_\nu \times \mathbf{r}_\tau|} dS$, $\mathbf{M}_2 = -\int_S \sigma_0 \left[\frac{\mathbf{r}_\tau \times (\mathbf{h}\cdot\boldsymbol{\omega}_\tau)}{|\boldsymbol{\omega}_\tau|}\cdot \mathbf{e}_3\right] \frac{\mathbf{r}_\tau \times (\mathbf{f}\cdot\mathbf{v}_S)}{|\mathbf{v}_S + \boldsymbol{\omega}_\nu \times \mathbf{r}_\tau|} dS$, $\mathbf{M}_3 = -\int_S \sigma_0 \frac{\mathbf{r}_\tau \times \left[\mathbf{f}\cdot(\boldsymbol{\omega}_\nu \times \mathbf{r}_\tau)\right]}{|\mathbf{v}_S + \boldsymbol{\omega}_\nu \times \mathbf{r}_\tau|} dS$, $\mathbf{M}_4 = -\int_S \sigma_0 \left[\frac{\mathbf{r}_\tau \times (\mathbf{h}\cdot\boldsymbol{\omega}_\tau)}{|\boldsymbol{\omega}_\tau|}\cdot \mathbf{e}_3\right] \frac{\mathbf{r}_\tau \times \left[\mathbf{f}\cdot(\boldsymbol{\omega}_\nu \times \mathbf{r}_\tau)\right]}{|\mathbf{v}_S + \boldsymbol{\omega}_\nu \times \mathbf{r}_\tau|} dS$. The vector $\mathbf{v}_S$ denotes the summary velocity in the point $M \in S$: $\mathbf{v}_S = \mathbf{v}_0 - R\boldsymbol{\omega}_\tau \times \mathbf{e}_3; \quad |\mathbf{v}_\Sigma|^2 = v_S^2 + 2\mathbf{v}_S \cdot (\boldsymbol{\omega}_\nu \times \mathbf{r}_\tau) + (\boldsymbol{\omega}_\nu \times \mathbf{r}_\tau)\cdot(\boldsymbol{\omega}_\nu \times \mathbf{r}_\tau)$.

Let us consider the orthotropic dry friction given by the tensor: $\mathbf{f} = f\begin{pmatrix}1 & 0\\ 0 & \kappa\end{pmatrix}, \quad f \neq 0, \quad \kappa \neq 0.$. Let us introduce the frame Ox^1x^2 attached to the centroid of the contact spot S; the corresponding base vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are collinear to the principal directions of the tensor $\mathbf{f}$. Let us consider the motion defined by the longitudinal velocity $\mathbf{v}_0 = V_0\mathbf{e}_1'$ along the axis OX^1 of the global rest frame, the rolling angular velocity $\boldsymbol{\omega}_\tau = -\omega_\tau \mathbf{e}_2$, and the spinning velocity ω_ν. Let us also consider here only circular area of contact with the radius R. Under these propositions the relationships are approximated by smooth analytical functions. The resultant force vector can be represented as $\mathbf{T} = T_\parallel \mathbf{e}_1 + T_\perp \mathbf{e}_2$, so that $T_\parallel$ is the longitudinal and $T_\perp$ is the lateral friction force. As a result, we have the following formulae: $T_\parallel = F_0 \frac{v_s}{\sqrt{v_s^2 + au^2}}$, $T_\perp = h\kappa F_0 \frac{uv_s}{\sqrt{(u^2 + bv_s^2)(v_s^2 + au^2)}}$, $M_\nu = M_0 \frac{u}{\sqrt{u^2 + mv_s^2}}$. Here $u = \omega_\nu R$, $F_0 = fN_0$ is the longitudinal resting friction force, but $M_0 = \pi(1+\kappa) f \int_0^1 \sigma_0(r) r^2 dr$ is the resting friction torque. Here the dimensionless radial coordinate is introduced. For the factors a, b, m we have following formulae analogous to these published in [1].

References

[1] Kireenkov A.A., Zhavoronok S.I. (2017) Coupled Dry Friction Models in Problems of Aviation Pneumatics' Dynamics, *Int. J. Mech. Sci.* DOI 10.1016/j.ijmecsci. 2017.02.004.