The Painlevé paradox and blowup - Part II

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Summary. We consider the problem of a rigid body, subject to a unilateral constraint, in the presence of Coulomb friction. We regularize the problem by assuming compliance (with both stiffness and damping) at the point of contact. Using a rigorous mathematical approach, we recover impact without collision (IWC) in both the inconsistent and indeterminate Painlevé paradoxes, in the latter case giving an exact formula for conditions that separate IWC and lift-off. We solve the problem for arbitrary values of the compliance damping and give explicit asymptotic expressions in the limiting cases of small and large damping.

Problem

In problems with unilateral constraints in the presence of friction, the rigid body assumption can result in the governing equations having multiple solutions (the *indeterminate* case) or no solutions (the *inconsistent* case). The classical example of Painlevé [9], consisting of a slender rod slipping along a rough surface (see Fig. 1), is the simplest and most studied example of these phenomena, now known collectively as *Painlevé paradoxes* [2, 3]. In Part I, we proved that a canard was present in the indeterminate case. Now in Part II, we consider the inconsistent case.

When a system is in an *inconsistent* state, it can not remain there. Lecornu [7] proposed a jump in vertical velocity to escape an inconsistent, horizontal velocity, state. This jump has been called *impact without collision* (IWC) [4]. IWC occurs instantaneously. So it must be incorporated into the rigid body formulation [6] by considering the equations of motion in terms of the normal impulse, rather than time. However, this process has been controversial [1], because it can sometimes lead to an apparent energy gain in the presence of friction. One way to address the Painlevé paradox is to *regularize* the rigid body formalism. Mathematically, very little rigorous work has been done on how IWC and Painlevé paradoxes can be regularized, until now.

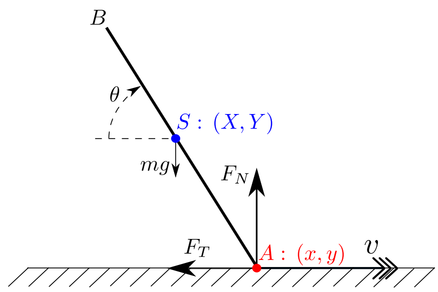


Figure 1: The classical Painlevé problem.

Governing equations

In Fig. 2, we show the (θ, ϕ) phase plane of the rigid body motion with the special point P [4] that divides the configurations of the rod into four quadrants. In this paper, we are interested in the configurations in the first (purple) and fourth (green) quadrants.

We assume that there is compliance at the point A between the rod and the surface, when they are in contact (see Fig. 1). Following [8], we assume that there are small excursions into $y < 0$. Then we take the nonnegative normal force $F_N(y, w)$ is a piecewise smooth function of (y, w): $F_N(y, w) = \epsilon^{-1}\left[-\epsilon^{-1}y - \delta w\right]$ where ϵ is a small parameter related to the spring constant, δ is the damping and the operation $[\cdot]$ is defined by

$$[f(y,w)] \equiv \begin{cases} 0 & \text{for} \quad y > 0 \\ \max\{f(y,w), 0\} & \text{for} \quad y \le 0, \end{cases} \tag{1}$$

The choice of scaling [8] ensures that the critical damping coefficient is independent of ϵ. Then we find

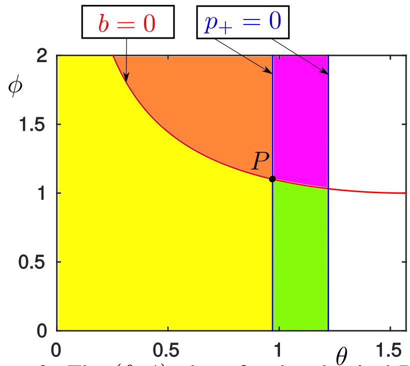


Figure 2: The (θ, ϕ)-plane for the classical Painlevé problem of Fig. 1.

$$\begin{aligned} \dot{x} &= y, & \dot{y} &= w, & \dot{w} &= b(\theta,\phi) + p_{\pm}(\theta)\epsilon^{-1}[-\epsilon^{-1}y - \delta w], \\ \dot{\theta} &= \phi, & \dot{\phi} &= c_{\pm}(\theta)\epsilon^{-1}[-\epsilon^{-1}y - \delta w], & \dot{v} &= a(\theta,\phi) + q_{\pm}(\theta)\epsilon^{-1}[-\epsilon^{-1}y - \delta w]. \end{aligned} \tag{2}$$

where $a(\theta, \phi), b(\theta, \phi), c_{\pm}(\theta), p_{\pm}(\theta), q_{\pm}(\theta)$ are lengthy problem-dependent functions.

Results

Theorem 1 shows that, if the rod starts in the fourth (green) quadrant of Fig. 2, it undergoes a (regularized) IWC for a time of $\mathcal{O}(\epsilon \ln \epsilon^{-1})$. The same theorem also gives expressions for the resulting vertical velocity of the rod in terms of the compliance damping and initial horizontal velocity and orientation of the rod.

Theorem 1 *Consider an initial condition*

$$(y, w, \theta, \phi, v) = (0, \mathcal{O}(\epsilon), \theta_0, \phi_0, v_0), \quad v_0 > 0, \tag{3}$$

within the green region of inconsistency (non-existence) where $p_+(\theta_0) < 0, b(\theta_0,\phi_0) < 0$ and $q_+(\theta_0) < 0$, $q_-(\theta_0) > 0$, $a \neq 0$. Then the forward flow of (3) *under* (2) *returns to* $\{(y,w,\theta,\phi,v)|y=0\}$ *after a time* $\mathcal{O}(\epsilon \ln \epsilon^{-1})$ *with*

$$w = e(\delta,\theta_0)v_0 + o(1), \qquad \phi = \phi_0 + \left\{-\frac{c_+(\theta_0)}{q_+(\theta_0)} + \frac{S_\phi(\theta_0)}{S_w(\theta_0)}\left(e(\delta,\theta_0) + \frac{p_+(\theta_0)}{q_+(\theta_0)}\right)\right\} v_0 + o(1), \tag{4}$$

and $\theta = \theta_0 + o(1), v = o(1)$ as $\epsilon \to 0$, where $S_{\phi,w}$ are known functions on $v = 0$, derived using the Filippov convention. During this time $y = \mathcal{O}(\epsilon)$, $w = \mathcal{O}(1)$ so that $F_N = \mathcal{O}(\epsilon^{-1})$. The function $e(\delta,\theta_0)$ is smooth and monotonic in δ and has the following asymptotic expansions:

$$e(\delta,\theta_0) = \frac{p_-(\theta_0) - p_+(\theta_0)}{q_-(\theta_0)p_+(\theta_0) - q_+(\theta_0)p_-(\theta_0)}\delta^{-2}\left(1 + \mathcal{O}(\delta^{-2}\ln\delta^{-1})\right) \quad \textit{for} \quad \delta \gg 1, \tag{5}$$

$$e(\delta,\theta_0) = \sqrt{\frac{p_+(\theta_0)(p_-(\theta_0) - p_+(\theta_0))}{q_+(\theta_0)(q_-(\theta_0) - q_+(\theta_0))}}\Bigg(1 - \frac{\sqrt{S_w(\theta_0)}}{2}\left(\pi - \arctan\left(\sqrt{-\frac{S_w(\theta_0)}{p_+(\theta_0)}}\right)\right)\delta + \mathcal{O}(\delta^2)\Bigg) \quad \textit{for} \quad \delta \ll 1. \tag{6}$$

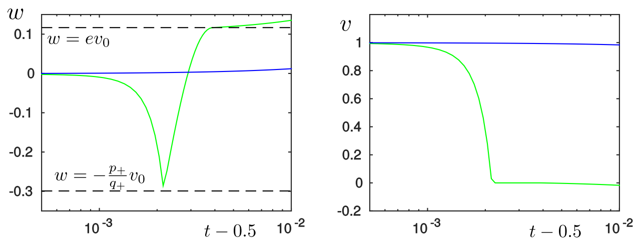
□

In Theorem 2, the rod starts in the first (purple) quadrant of Fig. 2.

Theorem 2 *Consider an initial condition $(y,w,\theta,\phi,v) = (0,\epsilon w_{10},\theta_0,\phi_0,v_0)$ for $w_{10} < w_{1*} \equiv -\lambda_-(\theta_0)\frac{b(\theta_0,\phi_0)}{p_+(\theta_0)} < 0$, where λ_- is a known function, within the purple region of indeterminacy (non-uniqueness) where $p_+(\theta_0) < 0, b(\theta_0,\phi_0) > 0$ and $q_+(\theta_0) < 0$, $q_-(\theta_0) > 0$, $a \neq 0$. Then the conclusions of Theorem 1 still hold true as $\epsilon \to 0$. For $w_{10} > w_{1*}$ lift-off occurs directly after a time $\mathcal{O}(\epsilon)$ with $w = \mathcal{O}(\epsilon)$. During this period $y = \mathcal{O}(\epsilon^2)$, so $F_N = \mathcal{O}(1)$.* □

The indeterminate case described by Theorem 2 is characterised by an extreme exponential splitting in phase space. In Fig. 3 we illustrate this for two rods (green and blue) initially ($t = 0$) distant by an amount of 10^{-3} above the compliant surface. At around $t = 0.5$, impact with the compliant surface occurs. The green rod experiences IWC whereas the blue rod lifts off directly. Fig. 3(a) and Fig. 3(b) show w and v as functions of time near $t = 0.5$.

Conclusions



(a) (b)

Figure 3: Dynamics of the Painlevé rod for $\epsilon = 10^{-3}$ in the indeterminate case.

We have considered the problem of a rigid body, subject to a unilateral constraint, in the presence of Coulomb friction. Our approach was to regularize the problem by assuming a compliance with stiffness and damping at the point of contact. This leads to a slow-fast system, where the small parameter ϵ is the inverse of the square root of the stiffness. The main achievement of this paper is to rigorously derive results that have eluded others in simpler settings. There are no existing results comparable to (5) and (6) for any value of δ. Our results are presented for arbitrary values of the compliance damping and we are able to give explicit asymptotic expressions in the limiting cases of small and large damping, all for a large class of rigid bodies, including the case of the classical Painlevé example in Fig. 1. We have been able to derive an explicit connection between the initial horizontal velocity of the body and its lift-off vertical velocity, for arbitrary values of the compliance damping, as a function of the initial orientation of the body. Our results can be generalised to a general class of rigid body and a general class of normal reaction [5].

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