Solution of Scale Dynamic Systems

Aftab Ahmed and Erik I. Verriest
Department of Electrical and Computer Engineering, Georgia Institute of Technology, U.S.A.

Summary. The main object of investigation in this paper is Scale Dynamic Systems (SDS). These systems form a typical class of time-varying delay differential equations where the delay is unbounded but the delay rate is less than or equal to unity. The basic form is $\dot{x}(t) = \pm x(\alpha t); \alpha \in [0, 1]$. We investigate this behavior for the scalar and higher order SDS in an operator theoretic framework, analyze the spectrum and give necessary and sufficient conditions for the asymptotic stability.

INTRODUCTION

Consider the following system characterized by the functional differential equation (FDE),

$$\dot{x}(t) = bx(\alpha t) \tag{1}$$

where $b \in \mathbb{R}$ and $\alpha \in [0, 1]$. The boundary (endpoint) cases i.e., $\alpha = 0$ and $\alpha = 1$ are trivial cases and are of least interest since in both the case the FDE reduces to an ODE. In the literature, such systems are also termed as scale delay systems [4]. Such systems appear in physics and engineering e.g., Chernekov radiation, light absorption by interstellar matter, collection of current by the pantograph of an electric locomotive, the theory of dielectric materials, number theory and continuum mechanics; see [5], [3], [4] and the references therein.
Fundamentally, this is a delay system with the unbounded time-variant delay $\tau(t)$ satisfying $\tau(t) = t - \alpha t = (1 - \alpha)t$ and $\dot{\tau}(t) = 1 - \alpha \le 1$. This bounded rate condition ($\dot{\tau} \le 1$) makes the time-varying delay system well-posed and causal. See [1] and [2] for the well-posedness associated with time-varying delay systems.
Now, we consider the following system.

$$\dot{x}(t) = ax(t) + bx(\alpha t) \tag{2}$$

The above equation is also called pantograph equation, see e.g., Iserles [3] and the references therein. Using Picard-Lindelöf method, the following series solution can be obtained for the SDS of (2).

$$x(t) = \left(1 + \sum_{n=1}^{\infty} \prod_{j=0}^{n-1} (a + b\alpha^j) \frac{t^n}{n!}\right) x(0) \tag{3}$$

Necessary & Sufficient Condition for Asymptotic Stability

Let us consider the general SDS as follows.

$$\dot{x}(t) = Ax(t) + Bx(\alpha t) \tag{4}$$

where $x(t) \in \mathbb{R}^n$ and $A, B \in \mathbb{R}^{n \times n}$. We define our new states as $y_1(t) := x(t)$, $y_2(t) := x(\alpha t)$, $y_3(t) = x(\alpha^2 t)$, ..., $y_N(t) = x(\alpha^{N-1} t)$,..... Therefore, the new higher order state space realization will be as follows.

$$\dot{\mathbf{y}} = \mathcal{A}\mathbf{y}$$

where $\mathbf{y} = \begin{pmatrix} y_1 & y_2 & y_3 & \cdots & y_N & \cdots \end{pmatrix}^\top$ and

$$\mathcal{A} = \begin{pmatrix} A & B & & & & & \\ & \alpha A & \alpha B & & & & \\ & & \alpha^2 A & \alpha^2 B & & & \\ & & & \alpha^3 A & \alpha^3 B & & \\ & & & & \ddots & \ddots & \\ & & & & & \alpha^{N-1} A & \alpha^{N-1} B \\ & & & & & \ddots & \ddots \end{pmatrix}.$$

Let $\mathcal{A}_N$ be the $N \times N$ truncated matrix of $\mathcal{A}$. Notice that the matrix $\mathcal{A}_N$ has a bidiagonal structure and is a special case of an upper triangular matrix. The leading or principal diagonal contains $\alpha^j A$ and the superdiagonal contains block matrix entries of the form. $\alpha^j B$.

Claim 1 $\mathcal{A} : l_2 \to l_2$ *is a compact operator on the Hilbert space.*

Proof: Consider the sequence of operators $A_N : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ which are essentially finite dimensional operators ($\mathbb{R}^{Nn \times Nn}$ matrices). Then it is easy to show that $\mathcal{A}_N \to \mathcal{A}$ as $N \to \infty$ in the operator norm induced by l_2 norm. Since A_N is compact, so is $\mathcal{A}$. ■

Since $\mathcal{A}$, is an infinite dimensional compact operator, we can show using Fredholm alternative that the accumulation point $\{0\}$ is indeed in its spectrum. Notice that the spectrum of $\mathcal{A}$ can be given as follows.

$$\text{Spec}(\mathcal{A}) = \bigcup_{j=0}^{\infty} \alpha^j \text{Spec}(A) \cup \{0\} \tag{5}$$

Now, we give the following result as a necessary and sufficient conditions for the asymptotic stability of (4).

Theorem 1 *The system characterized by the self-starting dynamics given by (4) is asymptotically stable only if the matrix A is Hurwitz.*

Conjecture: *The system characterized by SDS (4) is asymptotically stable if and only if the matrix A is Hurwitz and $r_\sigma(A) > r_\sigma(B)$ where $r_\sigma(A)$ denotes the spectral radius of A.*

Operator Theoretic Treatment

Now, we want to extend linear algebraic concepts to infinite dimensions and enter the regime of operator theory with a more rigorous treatment. Let $\mathbb{L}_2^n$ represent the Banach space of all the functions $f : \mathbb{R}_+ \to \mathbb{R}^n$ which are square integrable (Lebesgue integrable), i.e., $\int_0^\infty \|f(t)\|^2 dt$ is well defined and finite. We define the $\mathbb{L}_2$-norm as

$$\|f\|_{\mathbb{L}_2} \equiv \sqrt{\int_0^\infty \|f(t)\|_2^2 dt}. \tag{6}$$

The norm on the right hand side is the usual Euclidean norm.
Now, let's generalize this operator theoretic treatment to higher order systems. First, we define the two operators $\mathcal{P} : \mathbb{L}_2^n[0,1] \to \mathbb{L}_2^n[0,1]$ and $\mathcal{P}_\alpha : \mathbb{L}_2^n[0,1] \to \mathbb{L}_2^n[0,1]$ as follows.

$$\mathcal{P}x(t) = \int_0^t x(\theta)d\theta; \quad x(t) \in \mathbb{R}^n \tag{7}$$

$$\mathcal{P}_\alpha x(t) = \int_0^{\alpha t} x(\theta)d\theta; \quad x(t) \in \mathbb{R}^n \tag{8}$$

It can be shown that both $\mathcal{P}$ and $\mathcal{P}_\alpha$ are not only bounded but also compact on the Banach space $\mathbb{L}_2^n[0,1]$. From (4), we have,

$$\begin{aligned}
& x(t) = x(0) + A\int_0^t x(\theta)d\theta + \frac{1}{\alpha}B\int_0^{\alpha t} x(\theta)d\theta \\
\Leftrightarrow \quad & x(t) = x(0) + A\mathcal{P}x(t) + \frac{1}{\alpha}B\mathcal{P}_\alpha x(t) \Leftrightarrow x(t) = \left[I - \left(A\mathcal{P} + \frac{1}{\alpha}B\mathcal{P}_\alpha\right)\right]^{-1} x(0) \\
\Leftrightarrow \quad & x(t) = \sum_{n=0}^{\infty} \left(A\mathcal{P} + \frac{1}{\alpha}B\mathcal{P}_\alpha\right)^n x(0)
\end{aligned} \tag{9}$$

where the condition for the convergence of the composite Neumann series (9) is $\|A\mathcal{P} + \frac{1}{\alpha}B\mathcal{P}_\alpha\| < 1$. In general, the operators $\mathcal{P}$ and $\mathcal{P}_\alpha$ do not commute. Notice that,

$$\|A\mathcal{P} + \frac{1}{\alpha}B\mathcal{P}_\alpha\| \leq \|A\mathcal{P}\| + \|\frac{1}{\alpha}B\mathcal{P}_\alpha\| \leq \|A\|\|\mathcal{P}\| + \frac{1}{\alpha}\|B\|\|\mathcal{P}_\alpha\| = \frac{1}{\sqrt{2}}\|A\| + \frac{1}{\alpha}\|B\|\sqrt{\frac{\alpha}{2}} = \frac{1}{\sqrt{2\alpha}}\left(\sqrt{\alpha}\|A\| + \|B\|\right).$$

Hence, if, $\|A\mathcal{P} + \frac{1}{\alpha}B\mathcal{P}_\alpha\| < 1$, the solution to the initial value problem with $t_0 = 0$ only depends on $x(0)$, giving this a "self-starting character". This means that the dynamics behave as a finite dimensional system whose evolution only depends on the germ $x(0)$, thus building up its own history.

References

[1] E. I. Verriest, "Well-Posedness of Problems Involving Time-varying Delays," *Proceedings of the MTNS-2010, 5-9 July, 2010, Budapest, Hungary*, pp. 1203-1210, 2010.

[2] E. I. Verriest, "Inconsistencies in Systems with Time-Varying Delays and Their Resolution," *IMA Journal of Mathematical Control and Information*, vol. 28, pp. 147-162, 2011.

[3] A. Iserles, "On the generalized pantograph functional-differential equation" *European Journal of Applied Mathematics, Cambridge University Press*, vol. 04, no. 1, pp. 01-38, 1993.

[4] E. I. Verriest, "Robust stability, adjoints and LQ control of scale delay systems" *Decision and Control, 1999. Proceedings of the 38th IEEE Conference on*, vol. 01, pp. 209-214, 1999.

[5] T. Kato and J. B. McLeod, "The functional-differential equation $y'(x) = ay(\lambda x) + by(x)$" *Bulletin of the American Mathematical Society*, 77(6), pp. 891-937, 1971.