Slow-Fast Hamiltonian Systems: Dynamics and Bifurcations

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Summary. We present first geometrical tools for the study of slow-fast Hamiltonian systems on a smooth manifold. As a representative example of slow-fast system of the least dimension we discuss the dynamics of a Duffing type equation with slow periodically varying parameters such that related fast Hamiltonian systems with one degree freedom change periodically their phase portrait from the nonlinear oscillator with the unique center to the classical Duffing system with "figure eight" formed by two separatrices of a saddle, and back. Also some generalizations onto multidimensional cases will be discussed.

Slow-Fast Hamiltonian Systems

Slow-fast Hamiltonian systems are met very frequently in applications in different fields of science when a system under consideration has two essentially different time scales. These applications range from astrophysics, plasma physics and ocean hydrodynamics to molecular dynamics and the motion of magnetic waves in magnetic media. Usually these problems are given in coordinate form but there are cases when either the related symplectic form is nonstandard or the system under study is of a kind where the corresponding symplectic form has to be found, in particular, when we deal with the system on a manifold.

In more details, let a smooth bundle $p : M \to B$ be given where (M, ω) is a C^∞-smooth presymplectic manifold with a closed constant rank 2-form ω and (B, λ) is a smooth symplectic manifold with its symplectic 2-form λ. The 2-form ω is supposed to be compatible with the structure of the bundle, that is the bundle fibers are symplectic manifolds with respect to the 2-form ω and the distribution on M generated by kernels of ω is transverse to the tangent spaces of the leaves and the dimensions of the kernels and of the leaves are complementary. Then a symplectic structure $\Omega_\varepsilon = \omega + \varepsilon^{-1} p^*\lambda$ on M is defined [3] for any positive small ε, where $p^*\lambda$ is the lift of 2-form λ to M. Given a smooth Hamiltonian H on M one gets a slow-fast Hamiltonian system with respect to Ω_ε. In such a setting a slow manifold SM for this system is defined if fast Hamiltonian systems given at $\varepsilon = 0$ by the restriction of the Hamiltonian on symplectic leaves $p^{-1}(b)$, $b \in B$, with a symplectic form ω_b have equilibria. Assuming SM a smooth submanifold of M, a slow Hamiltonian flow on SM is defined. There are two interesting problems related with the study of a slow-fast Hamiltonian system for small $\varepsilon > 0$: 1) describe in some details the orbit behavior of the system near its slow manifold when this manifold consists of regular points w.r.t. the projection $p : SM \to B$; 2) describe some details of the transitory behavior near singular points of p where the restriction of $Dp|_{SM}$ has a nonzero kernel. Both these problems studied in many papers, in particular, see [1, 2, 3]. They concerned mainly of systems with one fast and one slow degrees of freedom (though in [1] the number of slow degrees of freedom can be an arbitrary integer). Here we try to extend some results onto the case when either fast or slow degrees of freedom equal to two.

An example: slow varying Duffing-type equation

As a representative example of a slow-fast Hamiltonian system of the least dimension we consider a periodic in time Hamiltonian system in one degree of freedom with a slow varying parameter, namely a Duffing type equation

$$\dot{x} = y,\ \dot{y} = -\sin\theta - x\cos\theta - x^3,\ \dot{\theta} = \varepsilon \tag{1}$$

with Hamiltonian

$$H = \frac{y^2}{2} + \frac{x^4}{4} + \frac{x^2}{2}\cos\theta + x\sin\theta.$$

This system demonstrates all types of the orbit behavior possible for an one and a half degree of freedom Hamiltonian system. In the talk we discuss this behavior using tools available by now in the theory of two dimensional symplectic diffeomorphisms. Of course, it is not possible to present completely rigorous explanations of the chaotic behavior observed under simulations in the system. No tools exist nowadays which allow to give a more or less satisfactory picture of the motion in the chaotic zones for a smooth symplectic 2-dimensional diffeomorphism.

The system under study is rather simple in its form, it is reversible in the phase space $\mathbf{R^2} \times \mathbf{S^1}$ and related fast systems have a minimally possible number of degenerate equilibria of simplest type (parabolic ones). The investigation allowed us to find for the related Poincaré map:

- The region where there is an eternal adiabatic invariant;
- A disk-shaped region where the dynamics is chaotic, Lyapunov's exponent calculated numerically appeared positive, this region contains infinitely many hyperbolic periodic orbits with the homo- heteroclinic tangles;
- Existence of relaxation symmetric periodic orbits which pass during their period a portion of time near the unstable hyperbolic part of the slow curve, like for canard periodic orbits, and the remaining time making fast oscillations with large amplitudes;

- Infinitely many bifurcations of symmetric periodic orbits of different types.

To investigate the dynamics various tools are used: the results of [1] on the almost integrable normal form for the Hamiltonian near its almost elliptic slow curve, theory of adiabatic invariance [6, **?**], results by Fenichel on the existence of hyperbolic slow manifold, blow up technique to represent the orbit transition near the disruption points, for the case of Hamiltonian system this is intimately connected with different solutions of the Painlevé-I equation.

Regular points of SM

Some generalizations of results found in [1, 3] will be presented in the talk. They concern the cases when dimension of M is 6 and the dimension of either bundle leaves or the base B is equal to four.
Consider first the case when $dim\, B = 2$ and $rank\, \omega = 4$. Since p is a submersion, all leaves of the bundle $p : M \to B$ are smooth 4-dimensional submanifolds and all fast systems (at $\varepsilon = 0$) for a given smooth Hamiltonian H are in two degrees of freedom.
Suppose $m \in M$ be an equilibrium of the fast system on the leaf $F_b = p^{-1}(b)$. We assume m be a simple equilibrium, that is the differential of the fast vector field at m is nondegenerate (no zero eigenvalues). Then there is a local disk SM through m of dimension $dim\, B = 2$ whose points are equilibria of the related fast Hamiltonian vector fields near m. SM transversely intersects leaves F_b for b close to $p(m)$ and the restriction of $p|_{SM} : SM \to B$ is a diffeomorphism. Thus points of SM are regular points for the restriction of projection p on SM. The fast Hamiltonian systems for $b \in B$ close to $p(m)$ depend smoothly on a 2-dimensional parameter b. Thus the related equilibria can undergo bifurcations generic for 2-parameter unfoldings. Among them the passage through Hamiltonian Hopf Bifurcation [4] is one of the most interesting locally. Moreover, in [7] a codimension 2 bifurcation was just studied. Since the study near point m is local it can be performed in some local Darboux coordinates (x, y, u, v), where $x, y \in \mathbb{R}^2$ and $\omega = dx \wedge dy$, $\lambda = du \wedge dv$. Then a system with Hamiltonian H (which may depend smoothly on ε) is written as

$$\dot{x} = H_y,\ \dot{y} = -H_x,\ \dot{u} = \varepsilon H_v,\ \dot{v} = -\varepsilon H_u.$$

Suppose now that at $m = (\mathbf{0}, \mathbf{0}, 0, 0)$ (without loss of generality) the fast system at $\varepsilon = 0$ has at $(\mathbf{0}, \mathbf{0})$ an equilibrium such that the eigenvalues of the linearized system are two double pairs $\pm i\gamma \neq 0$ with 2-dimensional Jordan blocks for each of them (nonsemisimple case). Generically, this bifurcation is of codimension one if some quantity A determined by terms of the third and fourth orders in the normal form does not vanish [4, 7]. But if $A = 0$ but the coefficient A_1 in the normal form of the sixth order does not vanish, then this bifurcation is of the codimension 2. This was stated and related bifurcations of the local orbit behavior were investigated in [7]. For the case of a slow-fast system the local parameters at $\varepsilon = 0$ are slow variables (u, v) varying near the point $p(m)$. Assume that $dH_m \neq 0$, that is m is not an equilibrium of the system for small $\varepsilon \neq 0$. Then under some conditions the system can be reduced via isoenergetic reduction to the nonautonomous Hamiltonian system in two and a half degrees of freedom. We shall discuss the reduction of this system to autonomous one up to exponentially small error if the system is analytic.

Singular points of SM

Now suppose the dimension $dim\, B = 4$ and $rank\, \omega = 2$. Here all leaves of the bundle (map p is a submersion) are 2-dimensional. If m is a simple equilibrium of the fast vector field on the leaf F_b, $b = p(m)$, then again one has a smooth 4-dimensional slow manifold SM through m. Now we assume a point m be an equilibrium of the fast system and for all $b \in B$ close enough to $p(m)$ the set of equilibria of the fast systems on leaves F_b compose a piece of a smooth submanifold SM containing m. In addition, we assume the restriction of Dp on $T_m SM$ have a nonzero kernel, i.e. m is a degeneration point for the map $p_r : SM \to B$. This is valid for the case when for the fast system on F_b, $b = p(m)$, point m is a parabolic equilibrium with the double zeroth eigenvalue, semisimple (Jordan form is zero matrix) with additional inequalities for the normal form of the third, fourth, fifth and sixth orders. Assume $dH_m \neq 0$, that is m is not an equilibrium for the slow-fast system as $\varepsilon \neq 0$. Then after isoenergetical reduction near m we get a nonautonomous Hamiltonian system. Some its solutions will be discussed along with their relations with solutions of the slow-fast system.

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