The Existence of Extremal Solutions for a Coupled System of Nonlinear Fractional Integro-Differential Equations

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Summary. In this paper, by means of upper and lower solutions, we develop monotone iterative method for the existence of extremal solutions for coupled system of nonlinear fractional integro-differential equations with advanced arguments. We illustrate this technique with the help of an example.

Introduction

Fractional calculus is a branch of mathematical analysis, that provides integrals and derivatives of any arbitrary order and due to their multiple applications in many areas of science and engineering has grown extensively.
The monotone iterative technique based on upper and lower solutions is a powerful tool for proving the existence of extremal solutions of nonlinear differential problems.
As far as we know, few authors have applied this technique to the system of nonlinear fractional differential equations.
In this study, we consider the existence of extremal solutions for the following system of nonlinear Riemann-Liouville fractional integro-differential equations with advanced arguments:

$$\begin{cases} (D^\alpha x(t))' = f(t, D^\alpha x(t), D^\alpha y(t), x(t), y(t), D^\beta x(t), Tx(t), Sy(t)), \\ (D^\alpha y(t))' = g(t, D^\alpha x(t), D^\alpha y(t), x(t), y(t), D^\beta x(t), Tx(t), Sy(t)), \\ D^\alpha x(0) = x^*, D^\alpha y(0) = y^*, \\ t^{1-\alpha} x(t)|_{t=0} = 0,\ t^{1-\alpha} y(t)|_{t=0} = 0,\ 0 < \beta \le \alpha \le 1, \end{cases} \tag{1}$$

where $t \in J := [0, T],\ f \in C(J \times \mathbb{R}^7, \mathbb{R})$,

$$(Tx)(t) = \int_0^t k(t,s)x(s)ds, \quad (Sy)(t) = \int_0^T h(t,s)y(s)ds.$$

Also $k(t,s) \in C[D, \mathbb{R}^+],\ h(t,s) \in C[[0,T]^2, \mathbb{R}^+],\ D = \{(t,s) \in \mathbb{R}^2 |\ 0 \le s \le t \le T\}$ and D^α, D^β are the Riemann-Liouville fractional derivatives.

Preliminaries

In this section, we present some definitions and results which will be needed later.

Lemma 0.1 *The coupled system of nonlinear fractional differential equation (1) is equivalent to the following initial value problem:*

$$\begin{cases} u'(t) = f(t, u(t), v(t), I^\alpha u(t), I^\alpha v(t), I^{\alpha-\beta} u(t), T_1 u(t), S_1 v(t)), \\ v'(t) = g(t, u(t), v(t), I^\alpha u(t), I^\alpha v(t), I^{\alpha-\beta} u(t), T_1 u(t), S_1 v(t)), \\ u(0) = x^*,\ v(0) = y^*,\ 0 < \beta \le \alpha \le 1,\ t \in J := [0, T], \end{cases} \tag{2}$$

where

$$T_1 u(t) = \int_0^t k_1(t,s)u(s)ds, \quad S_1 v(t) = \int_0^T h_1(t,s)v(s)ds,$$

$$k_1(t,s) = \int_s^t \frac{(\tau - s)^{\alpha-1} k(t,\tau)}{\Gamma(\alpha)} d\tau, \quad h_1(t,s) = \int_s^T \frac{(\tau - s)^{\alpha-1} h(t,\tau)}{\Gamma(\alpha)} d\tau.$$

Theorem 0.1 *Let the following assumptions hold:*

- *(H_1) There exist* $(u_0, v_0), (\alpha_0, \beta_0) \in C^1(J, \mathbb{R}) \times C^1(J, \mathbb{R})$ *such that* $\forall t \in J$ *satisfying* $(u_0(t), v_0(t)) \le (\alpha_0(t), \beta_0(t)),\ \Big(u_0(t) \le \alpha_0(t),\ v_0(t) \le \beta_0(t)\Big)$,

$$\begin{cases} u_0'(t) \le f(t, u_0(t), v_0(t), I^\alpha u_0(t), I^\alpha v_0(t), I^{\alpha-\beta} u_0(t), T_1 u_0(t), S_1 v_0(t)), \\ u_0(0) \le x^*, \\ v_0'(t) \le g(t, u_0(t), v_0(t), I^\alpha u_0(t), I^\alpha v_0(t), I^{\alpha-\beta} u_0(t), T_1 u_0(t), S_1 v_0(t)), \\ v_0(0) \le y^*,\ 0 < \beta \le \alpha \le 1,\ t \in J, \end{cases} \tag{3}$$

and

$$\begin{cases} \alpha_0'(t) \geq f(t, \alpha_0(t), \beta_0(t), I^{\alpha}\alpha_0(t), I^{\alpha}\beta_0(t), I^{\alpha-\beta}\alpha_0(t), T_1\alpha_0(t), S_1\beta_0(t)), \\ \alpha_0(0) \geq x^*, \\ \beta_0'(t) \geq g(t, \alpha_0(t), \beta_0(t), I^{\alpha}\alpha_0(t), I^{\alpha}\beta_0(t), I^{\alpha-\beta}\alpha_0(t), T_1\alpha_0(t), S_1\beta_0(t)), \\ \beta_0(0) \geq y^*,\ 0 < \beta \leq \alpha \leq 1,\ t \in J, \end{cases} \tag{4}$$

- *(H_2) There exist constants $M,\ N \geq 0$ such that*

$$f(t, u, v, I^{\alpha}u, I^{\alpha}v, I^{\alpha-\beta}u, T_1u, S_1v) - f(t, \bar{u}, v, I^{\alpha}\bar{u}, I^{\alpha}v, I^{\alpha-\beta}\bar{u}, T_1\bar{u}, S_1v) \\ \geq -M(u - \bar{u}),$$

where $u_0 \leq \bar{u} \leq u \leq \alpha_0,\ v_0 \leq v \leq \beta_0\ \forall t \in J$,

$$g(t, u, v, I^{\alpha}u, I^{\alpha}v, I^{\alpha-\beta}u, T_1u, S_1v) - g(t, u, \bar{v}, I^{\alpha}u, I^{\alpha}\bar{v}, I^{\alpha-\beta}u, T_1u, S_1\bar{v}) \\ \geq -N(v - \bar{v}),$$

where $v_0 \leq \bar{v} \leq v \leq \beta_0,\ u_0 \leq u \leq \alpha_0\ \forall t \in J$.

Then there exist monotone iterative sequences $\{(u_n, v_n)\}, \{(\alpha_n, \beta_n)\}$ which converge uniformly to the extremal solutions $(u_*, v_*), (\alpha^*, \beta^*)$ of (2), respectively, where $\{(u_n, v_n)\}, \{(\alpha_n, \beta_n)\}$ are defined by

$$u_n(t) = x^* e^{-\int_0^t M ds} + \int_0^t e^{-\int_s^t M d\tau}\Big[f\Big(s, u_{n-1}(s), v_{n-1}(s), I^{\alpha}u_{n-1}(s), I^{\alpha}v_{n-1}(s), \\ I^{\alpha-\beta}u_{n-1}(s), T_1u_{n-1}(s), S_1v_{n-1}(s)\Big) \\ + Mu_{n-1}(s)\Big]ds,$$

$$v_n(t) = y^* e^{-\int_0^t N ds} + \int_0^t e^{-\int_s^t N d\tau}\Big[g\Big(s, u_{n-1}(s), v_{n-1}(s), I^{\alpha}u_{n-1}(s), I^{\alpha}v_{n-1}(s), \\ I^{\alpha-\beta}u_{n-1}(s), T_1u_{n-1}(s), S_1v_{n-1}(s)\Big) \\ + Nv_{n-1}(s)\Big]ds,$$

$$\alpha_n(t) = x^* e^{-\int_0^t M ds} + \int_0^t e^{-\int_s^t M d\tau}\Big[f\Big(s, \alpha_{n-1}(s), \beta_{n-1}(s), I^{\alpha}\alpha_{n-1}(s), I^{\alpha}\beta_{n-1}(s), \\ I^{\alpha-\beta}\alpha_{n-1}(s), T_1\alpha_{n-1}(s), S_1\beta_{n-1}(s)\Big) \\ + M\alpha_{n-1}(s)\Big]ds,$$

$$\beta_n(t) = y^* e^{-\int_0^t N ds} + \int_0^t e^{-\int_s^t N d\tau}\Big[g\Big(s, \alpha_{n-1}(s), \beta_{n-1}(s), I^{\alpha}\alpha_{n-1}(s), I^{\alpha}\beta_{n-1}(s), \\ I^{\alpha-\beta}\alpha_{n-1}(s), T_1\alpha_{n-1}(s), S_1\beta_{n-1}(s)\Big) \\ + N\beta_{n-1}(s)\Big]ds,$$

also

$$(u_0, v_0) \leq (u_1, v_1) \leq ... \leq (u_n, v_n) \leq ... \leq (\alpha_n, \beta_n) \leq (\alpha_{n-1}, \beta_{n-1}) \leq ... \leq (\alpha_0, \beta_0).$$

Main Result

In this section, we prove the existence of extremal solutions of (1).
Let $C_{1-\alpha}(J, \mathbb{R}) = \{u \in C(0, T];\ t^{1-\alpha}u \in C(J, \mathbb{R})\}$ and $DC_{1-\alpha}(J, \mathbb{R}) = \{u \in C_{1-\alpha}(J, \mathbb{R});\ D^{\alpha}u \in C^1(J, \mathbb{R})\}$.

Theorem 0.2 *Assume that:*
(H_1') There exist $w_0=(w_1,w_2), z_0=(z_1,z_2)\in DC_{1-\alpha}(J,\mathbb{R})\times DC_{1-\alpha}(J,\mathbb{R})$ *such that* $D^\alpha w_1(t)\le D^\alpha z_1(t),\ D^\alpha w_2(t)\le D^\alpha z_2(t)$ *and* $w_0(t), z_0(t)$*, are lower and upper solutions of (1),*

$$\begin{cases}(D^\alpha w_1(t))'\le f(t,D^\alpha w_1(t),D^\alpha w_2(t),w_1(t),w_2(t),D^\beta w_1(t),Tw_1(t),Sw_2(t)),\\ D^\alpha w_1(0)\le x^*,\ t^{1-\alpha}w_1(t)|_{t=0}=0,\\ (D^\alpha w_2(t))'\le g(t,D^\alpha w_1(t),D^\alpha w_2(t),w_1(t),w_2(t),D^\beta w_1(t),Tw_1(t),Sw_2(t)),\\ D^\alpha w_2(0)\le y^*,\ t^{1-\alpha}w_2(t)|_{t=0}=0,\ 0<\beta\le\alpha\le 1,\end{cases}\tag{5}$$

and

$$\begin{cases}(D^\alpha z_1(t))'\ge f(t,D^\alpha z_1(t),D^\alpha z_2(t),z_1(t),z_2(t),D^\beta z_1(t),Tz_1(t),Sz_2(t)),\\ D^\alpha z_1(0)\ge x^*,\ t^{1-\alpha}z_1(t)|_{t=0}=0,\\ (D^\alpha z_2(t))'\ge g(t,D^\alpha z_1(t),D^\alpha z_2(t),z_1(t),z_2(t),D^\beta z_1(t),Tz_1(t),Sz_2(t)),\\ D^\alpha z_2(0)\ge y^*,\ t^{1-\alpha}z_2(t)|_{t=0}=0,\ 0<\beta\le\alpha\le 1,\end{cases}\tag{6}$$

(H_2') There exist constants $M,\ N\ge 0$ *such that*

$$\begin{cases}f(t,D^\alpha x(t),D^\alpha y(t),x(t),y(t),D^\beta x(t),Tx(t),Sy(t))\\ -f(t,D^\alpha\bar{x}(t),D^\alpha y(t),\bar{x}(t),y(t),D^\beta\bar{x}(t),T\bar{x}(t),Sy(t))\\ \ge -M(D^\alpha x(t)-D^\alpha\bar{x}(t)),\\ \end{cases}\tag{7}$$

where $D^\alpha w_1(t)\le D^\alpha\bar{x}(t)\le D^\alpha x(t)\le D^\alpha z_1(t),\ D^\alpha w_2(t)\le D^\alpha y(t)\le D^\alpha z_2(t)$*.*

$$\begin{cases}g(t,D^\alpha x(t),D^\alpha y(t),x(t),y(t),D^\beta x(t),Tx(t),Sy(t))\\ -g(t,D^\alpha x(t),D^\alpha\bar{y}(t),x(t),\bar{y}(t),D^\beta x(t),Tx(t),S\bar{y}(t))\\ \ge -N(D^\alpha y(t)-D^\alpha\bar{y}(t)),\end{cases}\tag{8}$$

where $D^\alpha w_2(t)\le D^\alpha\bar{y}(t)\le D^\alpha y(t)\le D^\alpha z_2(t),\ D^\alpha w_1(t)\le D^\alpha x(t)\le D^\alpha z_1(t)$*.*

Then there exist monotone iterative sequences $\{w_n=(w_1^n,w_2^n)\},\{z_n=(z_1^n,z_2^n)\}$ which converge uniformly to the extremal solutions $w_*=(w_{1_*},w_{2_*}),\ z^*=(z_1^*,z_2^*)$ of (1), respectively.

Example

In this section, in order to clarify the above-mentioned technique, we consider the following example which is appeared in the most applied problems in engineering sciences.

$$\begin{cases}(D^{\frac12}x(t))'=-(1+t)D^{\frac12}x(t)-(3+t^2)D^{\frac12}y(t)+t^2(y(t))^2\\ \qquad\qquad +\frac{t}{15}D^{\frac14}x(t)+\int_0^t tsx(s)ds-\int_0^1 sy(s)ds,\ t\in[0,1],\\ (D^{\frac12}y(t))'=-\frac{t}{2}D^{\frac12}x(t)-(1+t)D^{\frac12}y(t)-t^2x(t)+y(t)\\ \qquad\qquad -\frac12 D^{\frac14}x(t)+\int_0^1 s^2y(s)ds,\ t\in[0,1],\\ D^{\frac12}x(0)=0,\ \ t^{\frac12}x(t)|_{t=0}=0,\\ D^{\frac12}y(0)=0,\ \ t^{\frac12}y(t)|_{t=0}=0,\end{cases}\tag{9}$$

where $\alpha=\frac12,\beta=\frac14$. By easy computation, we have $M=2,\ N=\frac12$.
Now, take $w_0(t)=(0,0),\ z_0(t)=(1,1)$. It is easy to see that $w_0,\ z_0$ are lower and upper solutions of (9) and all the conditions of theorem (0.2) hold.
Thus there exist iterative sequences $\{w_n=(w_1^n,w_2^n)\},\{z_n=(z_1^n,z_2^n)\}$ which converge uniformly to the extremal solutions $w_*=(w_{1_*},w_{2_*}),\ z^*=(z_1^*,z_2^*)$ of (9), respectively.

References

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