Stability of Time-Delay Systems: From Integer-Order to Fractional-Order Systems

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Summary. With the rapid development of fractional calculus and active control techniques, stability of fractional-delay systems has been an increasing interest in engineering applications. For the local stability of an equilibrium of a linear dynamical system, the integral form of Mihkailov's criterion, Nyquist criterion, and Stepan-Hassard's criterion are probably among the most effective criteria. This talk firstly discusses some key issues in the applications of the above mentioned stability criteria, and then presents generalizations of Stepan-Hassard's criterion from RDDEs to NDDEs and to fractional NDDEs.

Fractional-Delay Systems

Over the past few decades, an increasing interest in the applications of fractional calculus has been drawn to the community of dynamics and control [1, 2, 3, 4]. It is well-recognized that fractional calculus is a better mathematical tool for modeling materials with memory than the classical calculus [2, 4]. The simplest fractional model, Scott-Blair's rule, which generalizes Hook's rule for elasticity and Newton's rule for viscosity, was successfully used not only to model viscoelastic materials, but also to model protein absorption in biology, learning and forgetting in psychology [4]. In vibration control, when the damping is described by using fractional-order derivative, the vibration system is governed by a differential system with fractional-order derivative. Unlike a viscous damping that can change the system stability only, a damping described by fractional-order derivative can change not only system stability but also vibration frequency simultaneously. In addition, fractional-order controllers extend the classical control theory. For example, the classical PID controller has been generalized to $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ with fractional-order λ and μ [1, 3].

Fractional systems are different from the ones with integer-order derivatives only. For instance, a linear ordinary differential equation with or without a delay can be expressed in terms of eigenfunctions, but the solution of a linear fractional system can be represented by an eigenfunction expansion determined by initial values plus an improper definite integral that is independent of initial values, and the improper definite integral disappears for integer-order systems [2]. Integer-order derivatives or integrals are uniquely defined, but fractional-order derivative has several definitions, including Riemann-Liouville's derivative, Caputo's derivative, Grünwald-Letnikov's derivative, and so on. The different fractional derivatives may not the same in quantity [1]. The local stability of both integer-order systems and fractional systems can be determined by the root location of their eigenfunctions, while the eigenfunction of a integer-order system is single-valued function, the eigenfunction of a fractional system is multi-valued function [3]. Thus, analysis on fractional-order systems must be carefully done.

Time delays exist commonly in control systems, and they usually come from controllers, filters and actuators [5, 6, 7]. For example, a digital controller uses sampled signals as control input, and it usually cannot take place at the same time when the signals are sampled. The gap between the time when the signals are sampled and the time when the controller works is a time delay. Another example is the delay in the use of accelerometers that are widely used in vibration control. The measured acceleration signals are often disturbed by high-frequency noise, and are required to be filtered out using low-pass filters. Due to the inevitable group delay caused by filters, the filtered acceleration signals may increase time delays compared to other signals like velocity or position. Time delay are also very common in human-interaction systems. A time-delay system is described by either neutral delay differential equations (NDDEs, for short) or retarded delay differential equations (RDDEs, for short), depending on whether the highest-order derivative of the controlled system has or has not delays respectively. For a system described by the mass-damper-spring model, for instance, the controlled system is a RDDE if a delayed PD controller is used, or a NDDE if a delayed acceleration feedback is introduced in the controller. Despite of its small value, time delay may destabilize the controlled system or deteriorate the system performance, as shown in many applications [7]. Time-delay systems have some special features different from the ones without time delays. In control applications, for example, positive feedback (feedback with positive gain) is seldom used, because it may enlarge state variation and deteriorate system stability. When the delay in control is taken into account, however, a delayed positive can be better than the corresponding negative feedback. As shown in the control of a SDOF oscillator [8, 9], for example, a delayed positive feedback can yield a much larger stable delay interval and the solution decays to zero faster than the corresponding negative feedback, and a delayed positive acceleration feedback can be well used to postpone the occurrence of Hopf bifurcation in delayed nonlinear oscillators [9]. Similar result is true for the problem of total absorption to a harmonic excited vibration by using a delayed resonator with acceleration feedback [10]. When a time delay is involved in a fractional-order system, the system becomes a fractional-delay system, which can be either a RDDE with fractional-order derivative or a NDDE with fractional-order derivative, depending on whether the highest-order derivative of the controlled system has or has not delays respectively.

Stability Analysis

In a sense, the stability of linear fractional-order systems can be studied similarly as done for integer-order systems. While the solution of a linear system of integer-order is in terms of eigenfunctions using exponential functions, the solution of a linear fractional system can be represented by using Mittag-Leffler functions, which are generalizations of exponential functions. However, compared with the exponential functions that have clear geometrical interpretation, the infinite series expansions of Mittag-Leffler functions are much harder to be understood. In addition, as mentioned above, the solution of a fractional system can be represented by an eigenfunction expansion determined by initial values plus an improper integral independent of initial values [2], and the improper integral decays rapidly to zero as the time goes to infinity. Thus, the eigenfunction expansion using exponential functions consists of the dominant part of the solution of a fractional system. As a results, the stability of a fractional system can be studied similarly as done for integer-order systems, using exponential functions. In this way, for the local stability of an equilibrium, the key step is to check whether the roots of the single-valued eigenfunction (or in term, characteristic function) have negative real parts only.

For this purpose, the methods or criteria established on the basis of Argument Principle or equivalently Cauchy Theorem are shown helpful in applications. Three of such criteria, the integral form of Mihkailov's criterion, Nyquist criterion, and Stepan-Hassard's criterion, are probably among the most effective criteria for the stability analysis of time-delay systems. The integral form of Mihkailov's criterion has been generalized to fractional RDDEs [11] and fractional NDDEs [12], and the Nyquist criterion has been extended to fractional RDDEs in [13]. Stepan's criterion is a formula for calculating the unstable unstable characteristic roots (the characteristic roots with positive real parts) for RDDEs [5]. Stepan's formula was originally represented in different form for two cases: the order of the RDDE is odd, or the order of the RDDE is even. Hassard gave an unified form of Stepan's formula in [14]. A generalization of Hassard's criterion from RDDEs to fractional RDDEs was made in [15].

This talk firstly discusses some key issues in the applications of the above mentioned stability criteria, and then presents generalizations of Stepan-Hassard's criterion from RDDEs to NDDEs and to fractional NDDEs.

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