A Feasible Analysis of Quasi-Periodic Mathieu Equations via Floquét Theory

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Summary. In this work an approximate approach is proposed to determine the stability and response of quasi-periodic (QP) Mathieu type equations. Although Floquét theory is applicable only to periodic systems, it is suggested here that a QP system may be replaced by a periodic system with an approximate large principal period and thus making it suitable for an application of the Floquét theory. Based on this premise, a systematic approach has been developed and applied to a typical QP system. The approximate stability chart shows excellent agreement with numerical results.

Introduction

Parametrically excited systems are encountered in various fields of science and engineering. The mathematical models of these systems are generally represented by linear ordinary differential equations with time varying coefficients. In most cases, these systems have been modeled by Mathieu's or Hill's equations (periodic coefficients) because their stability and response can be determined using Floquét theory [1]. However, in reality, the parametric excitation is not periodic but consists of frequencies that are incommensurate; hence making it quasi-periodic (QP). For instance, parametric resonance in sailing ships has been studied via Mathieu/Hill equations. However, sea waves are not periodic but composed of incommensurate frequencies leading to a QP system. Beating heart is another example where the motion is also not purely periodic and the equation of motion turns out to be a QP Mathieu equation. Unfortunately, there is no complete theory for linear ordinary differential equations with QP coefficients. Therefore, in the present investigation an attempt has been made to provide an approximate methodology for a general class of QP systems that arises in science and engineering. The development of such an approximate theory allows one to construct Lyapunov-Perron transformation matrices that reduce the linear QP systems to the systems with constant coefficients.

Proposed Methodology

The damped QP Mathieu equation can be written in the state space form as,

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\left(a + \sum_{i=1}^{m} b_i \cos(\omega_i t)\right) & -d \end{bmatrix} x \qquad \text{or} \qquad \dot{x} = A(t)x \tag{1}$$

where, $x = \{x_1 \quad x_2\}^T \in \mathbb{R}^2$, $\omega = \{\omega_1, \omega_2, \ldots, \omega_m\} \in \mathbb{R}^m$ is the frequency basis of the coefficient matrix $A(t)$, a, b_i and d are the system parameters and t is the time. When $A(t)$ has finite ($m \geq 2$) incommensurate frequencies, $A(t)$ is QP. Floquét theory cannot be applied to determine the stability and response of Eq. (1) as the principal period of $A(t)$ tends to infinity. However, an approximate period can always be defined such that for every $\varepsilon > 0$, there exists a length of time $T(\varepsilon)$ that contains a number, $T_a(\varepsilon)$ for which $|A(t + T_a(\varepsilon)) - A(t)| < \varepsilon$. The $T_a(\varepsilon)$ can be determined by truncating the frequency module i.e. $|k_1\omega_1 \pm k_2\omega_2 \pm \cdots \pm k_m\omega_m|$ of $A(t)$ where $k_i = 0,1,2,\ldots$ and $k_1 = k_2 = \cdots = k_m \neq 0$. The '*minimum frequency*', $\omega_{\min}$ in the truncated frequency module is used to determine the approximate period of the QP system. Thus,

$$\omega_{\min} = Min(|k_1\omega_1 \pm k_2\omega_2 \pm \cdots \pm k_m\omega_m|) \quad ; \quad \omega_{\min} \neq 0 \qquad \text{and} \qquad T_a = 2\pi/\omega_{\min} \tag{2}$$

Once $\omega_{\min}$ and T_a have been determined, the original QP system can be replaced by following periodic system,

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\left(a + \sum_{i=1}^{m} b_i \cos(\bar{\omega}_i t)\right) & -d \end{bmatrix} x \tag{3}$$

where, ω_i is approximated by $\bar{\omega}_i$ such that $\bar{\omega}_i$ are the integral multiples of $\omega_{\min}$ and $\omega_{\min}$ remains the minimum frequency in the approximated system. Since the approximate system is periodic in nature, its stability and response can be determined by Floquét theory. It is expected that the approximate system (c.f. Eq. (3)) yields an approximate solution of the original QP system, Eq. (1). In the following, a two frequency case ($m = 2$ in Eq. (1)) is discussed to show the viability of the proposed method. For the sake of convenience numerical computations are performed by setting $b_1 = b_2 = b$. All computations are numerical and a, b and d are not assumed to be small.

A Case Study

Two frequency cases are the simplest examples of QP Mathieu equations. We consider the case of $\omega_1 = \pi$ and $\omega_2 = 7.0$. As a first step, $\omega_{\min}$ and T_a are calculated over a range of k_1 and k_2 and subsequently approximate system is determined by replacing ω_i in Eq. (1) by $\bar{\omega}_i$. Once the approximate system has been defined, stability charts can be plotted using the Floquét theory. Zounes and Rand [2] used the concept of winding number and reported that all the unstable regions in the QP systems arise from $a = (k_1\omega_1 \pm k_2\omega_2)^2/4$; $k_i = 0,1,2,\ldots$ and the main instability regions will stem from $a = \omega_1^2/4 = 2.46740$ ($k_1 = 1$ and $k_2 = 0$) and $a = \omega_2^2/4 = 12.25$ ($k_1 = 0$ and $k_2 = 1$) which are same as the bifurcation points of the main instability regions of periodic systems ($a = \omega_i^2/4$) with excitation frequencies $\omega_1 = \pi$ and $\omega_2 = 7.0$. The convergence study of bifurcation points of main instability regions may be used as a guideline in the selection of $\omega_{\min}$ and is shown in Fig. 1. It is expected that with relatively smaller $\omega_{\min}$, the proposed method will yield better results. However, with the decrease in $\omega_{\min}$, T_a increases and hence require longer computation time. Keeping computational time reasonable, $\omega_{\min} = 0.0442270$ is used in further investigations. For $\omega_{\min} = 0.0442270$, $T_a = 142.067$, $\bar{\omega}_1 = 71\omega_{\min}$ ($\approx \pi$) and $\bar{\omega}_2 = 158\omega_{\min}$ (≈ 7).
Fig. 2 shows the stability charts for the undamped system ($d = 0$). A number of T_a and $2T_a$ unstable regions could be observed in the $a \sim b$ plane. These unstable regions are either due to primary frequencies ($\bar{\omega}_1$ and $\bar{\omega}_2$) or due to various combinations of $\bar{\omega}_1$ and $\bar{\omega}_2$. Since the approximate system is periodic with period T_a, the unstable regions will originate from $a = (k\omega_{\min})^2/4$ in the $a \sim b$ plane and the main instability regions will stem from $a = 2.46508$ and $a = 12.2076$. These bifurcation points are close to the results obtained from Zounes and Rand [2] expression for bifurcation points. In fact, Zounes and Rand [2] expression is same as $a = (k\omega_{\min})^2/4$ if $(k_1\omega_1 \pm k_2\omega_2)$ is replaced by $(k\omega_{\min})$. Since k_1 and k_2 are finite by the definition of $\omega_{\min}$, the results obtained by the proposed method is a subset of results given by Zounes and Rand [2] expression.
The qualitative behavior of the approximate and exact solutions is studied with the help of Poincaré maps at some typical points near the stability boundaries of prominent regions (marked as R1 through R7 in Fig. 2) and are found to be similar. The Discrete Fourier Transform of the solutions ensured that the approximate stability boundaries computed using Floquét theory are accurate representation of exact boundaries.

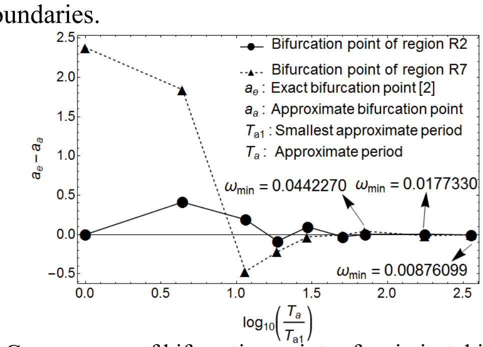

Figure 1: Convergence of bifurcation points of main instability regions.

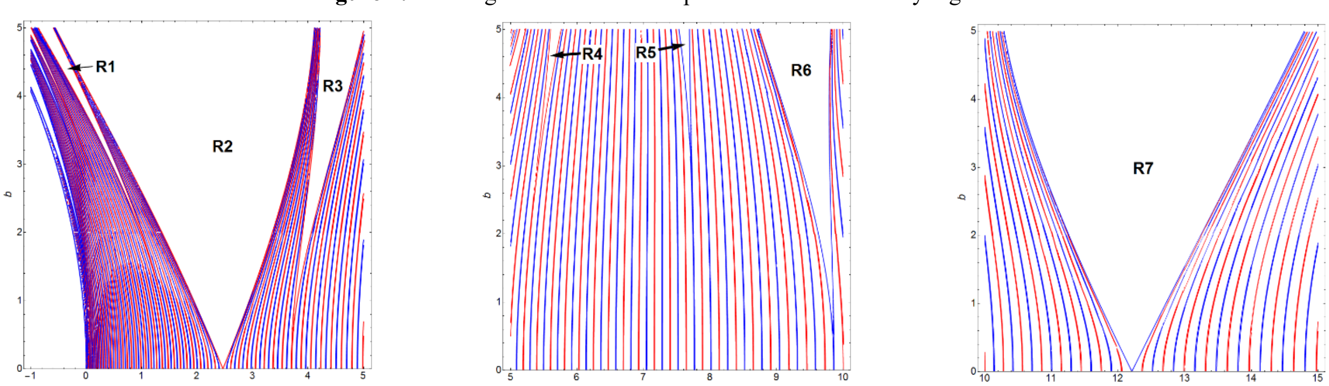

Figure 2: Stability charts for undamped system ($d = 0$): Red- $2T_a$ periodic, Blue- T_a periodic

References

[1] Grimshaw, R., 1990, *Nonlinear Ordinary Differential Equations*, Blackwell Scientific Publications, Massachusetts.

[2] Zounes, R.S., and Rand, R.H., 1998, "Transition Curves for the Quasi-Periodic Mathieu Equation," SIAM J. Appl. Math., **58**(4), pp. 1094-1115.